First Shifting property:

If $L\{f(t)\} = F(s)$ then

(i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

(ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof:

(i) We know that,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

(ii) $$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

Second Shifting property:

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$.

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t - a = u \Rightarrow dt = du$

When $t = a \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\,dt \quad \text{Replace } u \to t$$

$$L[g(t)] = e^{-as} F(s).$$

(1) Find $L\left[e^{-3t}\sin^2 t\right]$

Proof:

$$L\left[e^{-at}f(t)\right] = F(s+a)$$

$$L\left[e^{-3t}\sin^2 t\right] = L\left[\sin^2 t\right]_{s\to s+3}$$

$$= L\left[\frac{1-\cos 2t}{2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left\{L(1) - L(\cos 2t)\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{(s+3)\left[(s+3)^2+4\right]}\right\}$$

$$= \frac{2}{(s+3)\left[(s+3)^2+4\right]}$$

(1) Find $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= 0 + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{3e^{-2s}}{s}$$