Initial value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\underset{t \to 0}{Lt} [f(t)] = \underset{s \to \infty}{Lt} [s F(s)]$$

Proof:

We know that

$$L[f'(t)] = s L[f(t)] - f(0)$$

$$= s F(s) - f(0)$$

$$\Rightarrow s F(s) = L[f'(t)] + f(0)$$

$$s F(s) = \int_0^\infty e^{-st} f'(t)\, dt + f(0).$$

Taking limit as $s \to \infty$ on both sides we get,

$$\underset{s \to \infty}{Lt} s F(s) = \underset{s \to \infty}{Lt} \left\{ \int_0^\infty e^{-st} f'(t)\, dt + f(0) \right\}$$

$$= \underset{s\to\infty}{Lt} \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \underset{s\to\infty}{Lt}\, e^{-st} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \underset{t\to 0}{Lt}\, f(t)$$

Hence $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$.

Final value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and $L[f(t)] = F(s)$ then

$$\underset{t\to\infty}{Lt}\,[f(t)] = \underset{s\to 0}{Lt}\,[sF(s)]$$

Proof:

We know that,

$$L[f'(t)] = sL[f(t)] - f(0)$$
$$= sF(s) - f(0)$$
$$\Rightarrow sF(s) = L[f'(t)] + f(0)$$

Taking limit $s\to 0$ on both sides, we get,

$$\underset{s\to 0}{Lt}\,[sF(s)] = \underset{s\to 0}{Lt}\left\{\int_0^\infty e^{-st} f'(t)\,dt + f(0)\right\}$$

$$= \int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0) = \underset{t\to\infty}{Lt}\, f(t)$$

Hence $\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\,[sF(s)]$

(1) Verify the initial and final value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln:

$$F(s) = L[1 + e^{-t}\sin t + e^{-t}\cos t]$$

$$= L(1) + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$$

$$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \to s+1} + \left(\frac{s}{s^2+1}\right)_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$\therefore\ s\,F(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

Initial value theorem: $\boxed{\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s.F(s)}$

$$\underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + 1 = 2$$

$$\underset{s\to\infty}{Lt}\ s\,F(s) = \underset{s\to\infty}{Lt}\ s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\ 1 + \frac{s^2+2s}{s^2+2s\left(1+\frac{2}{s^2+2s}\right)} \qquad = \underset{s\to\infty}{Lt}\ 1 + \frac{s^2\left[1+\frac{2}{s}\right]}{s^2\left[1+\frac{2}{s}\left(1+\frac{2}{s^2+2s}\right)\right]}$$

$$= 1 + 1$$

$$= 2$$

Hence $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s\,F(s) = 2$

IVT is verified.

Final value theorem: $\boxed{\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s.F(s)}$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \lim_{s\to 0}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

Hence $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 1$, FVT is Verified

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{-1}{s}\left(e^{-st}\right)_0^b + \frac{1}{s}\left(e^{-st}\right)_b^{2b}\right]$$

$$= \frac{1}{s(1-e^{-2bs})}\left[-(e^{-bs}-1) + (e^{-2bs}-e^{-bs})\right]$$

$$= \frac{-e^{-bs}+1+(e^{-bs})^2-e^{-bs}}{s(1-e^{-2bs})} = \frac{1-2e^{-bs}+(e^{-bs})^2}{s(1-e^{-2bs})}$$

$$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})}(1-e^{-bs})^2$$

$$= \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$= \frac{1}{s}\left(\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right)$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right)$$

(2) Find the Laplace transform of the half wave rectifier function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Soln:

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + 0 \right]$$

Formula

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

Here

$x = t$

$a = -s$

$b = \omega$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega} \cdot \omega + \omega}{s^2+\omega^2} \right]$$

$$= \frac{\omega \left[\cancel{1+e^{-s\pi/\omega}} \right]}{(1-e^{-s\pi/\omega})\cancel{(1+e^{-s\pi/\omega})}(s^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$