Inverse Laplace Transform:

If the Laplace transform of $f(t)$ is $F(s)$ i.e., $L[f(t)] = F(s)$. Then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is written as $f(t) = L^{-1}[F(s)]$ where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace Transforms:

	$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
1	$L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
2	$L(t) = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
3	$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
4	$L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
5	$L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
6	$L(\sin at) = \frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
7	$L\left(\frac{\sin at}{a}\right) = \frac{1}{s^2+a^2}$	$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$
8	$L(\cos at) = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
9	$L(\sinh at) = \frac{a}{s^2-a^2}$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
10	$L(\cosh at) = \frac{s}{s^2-a^2}$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
11	$L[\delta(t)] = 1$	$L^{-1}(1) = \delta(t)$

Problems :

(1) Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Soln:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+b^2} - \frac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left(\frac{1}{s^2+b^2}\right) - a^2 L^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right)$$

$$= \frac{1}{b} L^{-1}\left(\frac{b}{s^2+b^2}\right) - \frac{a^2}{b^2-a^2} L^{-1}\left(\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right)$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2} L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

(2) Find $L^{-1}\left[\frac{2s-5}{9s^2-25}\right]$

Soln:

$$L^{-1}\left(\frac{2s-5}{9s^2-25}\right) = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right]$$

$$= L^{-1}\left[\frac{2s}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]} - \frac{5}{9\left[s^2-\left(\frac{5}{3}\right)^2\right]}\right]$$

$$= \frac{2}{9} L^{-1}\left[\frac{s}{s^2-\left(\frac{5}{3}\right)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{5/3}{s^2-(5/3)^2}\right]$$

$$= \frac{2}{9}\cosh\frac{5}{3}t - \frac{1}{3}\sinh\frac{5}{3}t$$

(3) Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$

Soln:

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \frac{d}{dt}\left[L^{-1}\left(\frac{1}{(s+2)^2+4}\right)\right]$$

$$= \frac{d}{dt}\left[e^{-2t}L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}\sin 2t\right) = \frac{1}{2}\frac{d}{dt}\left(e^{-2t}\sin 2t\right)$$

$$= \frac{1}{2}\left[e^{-2t}\cos 2t\,(2) + \sin 2t\,(-2)\,e^{-2t}\right]$$

$$= \frac{1}{2}\left[2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t\right]$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$