Applications of Laplace transforms to Differential equations:

If $L[f(t)] = F(s)$ then

$$L[y'(t)] = sL(y) - y(0)$$

$$L[y''(t)] = s^2 L(y) - sy(0) - y'(0).$$

① Solve the differential equations using LT $y'' + 4y' + 4y = e^{-t}$ given that $y(0) = 0$ and $y'(0) = 0$.

Soln:

$$y'' + 4y' + 4y = e^{-t}$$

Taking LT on both sides,

$$L(y'' + 4y' + 4y) = L(e^{-t})$$

$$L(y'') + 4L(y') + 4L(y) = \frac{1}{s+1}$$

$$[s^2 L(y) - sy(0) - y'(0)] + 4[sL(y) - y(0)] + 4L(y) = \frac{1}{s+1}$$

Given: $y(0)=0, \quad y'(0)=0$

$$\Rightarrow [s^2 L(y) - s\times 0 - 0] + 4[s L(y) - 0] + 4L(y) = \frac{1}{s+1}$$

$$\Rightarrow s^2 L(y) + 4s L(y) + 4L(y) = \frac{1}{s+1}$$

$$\Rightarrow (s^2+4s+4) L(y) = \frac{1}{s+1}$$

$$\Rightarrow L(y)(s+2)^2 = \frac{1}{s+1}$$

$$L(y) = \frac{1}{(s+1)(s+2)^2}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

Put $s=-2 \Rightarrow C=-1$

Put $s=-1 \Rightarrow A=1$

Put $s=0 \Rightarrow B=-1$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= e^{-t} - e^{-2t} - te^{-2t}$$

$$L^{-1}\left(\frac{1}{(s+2)^2}\right) = e^{-2t} \cdot L^{-1}\left(\frac{1}{s^2}\right) = e^{-2t} \cdot t$$