Transforms of elementary functions:

1) $L(1) = \frac{1}{s}$ where $s>0$.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(1) = \int_0^\infty e^{-st}.1\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$L(1) = \frac{1}{s}$$

2) $L(K) = \frac{K}{s}$.

3) $L(t) = \frac{1!}{s^2}$

Bernoulli's formula:

$$I = uv_1 - u'v_2 + u'''v_3$$

$u = t$, $u' = 1$, $u'' = 0$

$v = e^{-st}$, $v' = \frac{e^{-st}}{-s}$, $v_2 = \frac{e^{-st}}{s^2}$

$$L(t) = \int_0^\infty e^{-st}.t\,dt$$

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$L(t) = \frac{1!}{s^2}$$

4) $L(t^2) = \frac{2!}{s^3}$

5) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ if $s>0$ and $n>-1$.

$$L(t^n) = \int_0^\infty e^{-st}\cdot t^n\,dt$$

Put $x = st$, $dx = s\,dt$

$$\frac{dx}{s} = dt$$

$$L(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x}\cdot\frac{x^n}{s^{n+1}}\,dx$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\,dx$$

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

6) $L(e^{at}) = \frac{1}{s-a}$ if $s-a>0$.

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$L(e^{at}) = \frac{1}{s-a}$ if $s-a>0$.

7) $L(e^{-at}) = \frac{1}{s+a}$ if $s+a>0$.

$$L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$L(e^{-at}) = \frac{1}{s+a} \text{ if } s+a>0.$$

8) To find $L(\cos at)$ and $L(\sin at)$.

We know $e^{i\theta} = \cos\theta + i\sin\theta$

$$L(e^{iat}) = \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$L(\cos at + i\sin at) = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating real & imaginary Parts,

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

9) To find $L(\sinh at)$:-

$$L[\sinh at] = L\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right)$$

$$L(\sinh at) = \frac{a}{s^2-a^2} \quad \text{for } s^2 > a^2$$

10) To find $L(\cosh at)$:-

$$L(\cosh at) = L\left(\frac{1}{2}(e^{at}+e^{-at})\right)$$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\,\frac{2s}{s^2-a^2}$$

$$L(\cosh at) = \frac{s}{s^2-a^2} \quad \text{for } s^2 > a^2$$

Problems :-

1) Find $L(t^8)$.

Sol :- $$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(t^8) = \frac{8!}{s^{8+1}} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{s^9} = \frac{46320}{s^9}$$

2) Find $L(t+1)$.

Sol: $L(t+1)^2 = L(t^2+2t+1)$

$$= L(t^2) + 2L(t) + L(1)$$

$$= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

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3) Find $L\left(\frac{1}{\sqrt{t}}\right)$.

Sol: $L\left(\frac{1}{\sqrt{t}}\right) = L\left(t^{-1/2}\right)$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\boxed{\begin{array}{l}\Gamma n+1 = n\Gamma n \\ \Gamma 1/2 = \sqrt{\pi}\end{array}}$$

4) $L(\sqrt{t})$:-

Sol: $L(\sqrt{t}) = L(t^{1/2})$

$$= \frac{\Gamma 1/2+1}{s^{1/2+1}} = \frac{\frac{1}{2}\Gamma 1/2}{s\sqrt{s}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

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5) $L(t^{5/2})$.

Sol: $L(t^{5/2}) = \frac{\Gamma \frac{5}{2}+1}{s^{5/2+1}} = \frac{\frac{5}{2}\Gamma\frac{5}{2}}{s^{7/2}}$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}}$$

$$= \frac{15\sqrt{\pi}}{8s^{7/2}}$$

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6) $L(e^{5t})$.

Sol: $L(e^{at}) = \frac{1}{s-a}$

$L(e^{5t}) = \frac{1}{s-5}$

7) $L(e^{t})$.

Sol: $L(e^{t}) = \frac{1}{s-1}$

8) $L(e^{-7t})$.

Sol: $L(e^{-7t}) = \frac{1}{s-(-7)} = \frac{1}{s+7}$

9) $L(e^{-t})$.

Sol: $L(e^{-t}) = \frac{1}{s-(-1)} = \frac{1}{s+1}$

10) Find $L(\sin 5t)$

$L(\sin at) = \frac{a}{s^2+a^2}$

$L(\sin 5t) = \frac{5}{s^2+5^2} = \frac{5}{s^2+25}$

11) Find $L(\cos 6t)$.

$L(\cos at) = \frac{s}{s^2+a^2}$

$L(\cos 6t) = \frac{s}{s^2+6^2}$

$= \frac{s}{s^2+36}$

12) Find $L(\sin^2 2t)$.

Sol: $\sin^2 2t = \frac{1-\cos 2t}{2}$

$$L(\sin^2 2t) = L\left(\frac{1-\cos 2(2t)}{2}\right)$$

$$= \frac{1}{2} L(1-\cos 4t)$$

$$= \frac{1}{2}\left[L(1) - L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

13) Find $L(\cos^2 3t)$.

Sol: $\cos^2 t = \frac{1+\cos 2t}{2}$

$$L(\cos^2 3t) = L\left(\frac{1+\cos 2(3t)}{2}\right)$$

$$= \frac{1}{2} L(1+\cos 6t)$$

$$= \frac{1}{2}\left[L(1) + L(\cos 6t)\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right)$$

14) Find $L(\cos^3 2t)$.

Sol: $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

$$L(\cos^3 2t) = L\left(\frac{\cos 3(2t) + 3\cos 2t}{4}\right)$$

$$= \frac{1}{4}\left\{L(\cos 6t) + 3L(\cos 2t)\right\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+36} + 3\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+36} + 3\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right\}$$

15) Find $L(\sin^3 3t)$.

Sol:. $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

$$= L\left(\frac{3\sin 3t - \sin 3(3t)}{4}\right)$$

$$= \frac{1}{4}\left\{3L(\sin 3t) - L(\sin 9t)\right\}$$

$$= \frac{1}{4}\left\{3\left(\frac{3}{s^2+3^2}\right) - \frac{9}{s^2+9^2}\right\}$$

$$= \frac{9}{4}\left\{\frac{1}{s^2+9} - \frac{1}{s^2+81}\right\}$$

16) Find $L(\sin 2t \cos 3t)$

Sol:. $\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$

$$L(\sin 2t \cos 3t) = L\left\{\frac{\sin(2t+5t) + \sin(2t-5t)}{2}\right\}$$

$$= \frac{1}{2}\left\{L(\sin 5t) + L(\sin(-t))\right\}$$

$$= \frac{1}{2}\left\{L(\sin 5t) - L(\sin t)\right\}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right\}$$