Integral of Laplace Transform (or) Laplace transform of $\frac{f(t)}{t}$:

If $L[f(t)] = F(s)$ and if $\lim_{t \to 0} \frac{f(t)}{t}$ exists then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Proof:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Integrating w.r.to 's' from s to ∞, we get

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,ds\right] dt$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-s}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$$

Problems:-

1) Find $L\left(\frac{1-\cos t}{t}\right)$

Sol:- $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty L(1-\cos t)\, ds$

$$= \int_s^\infty \{L(1) - L(\cos t)\} ds$$

$$= \int_s^\infty \left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+1)\right)_s^\infty$$

$$= \left(\log s - \log(s^2+1)^{1/2}\right)_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left[\frac{s}{\sqrt{s^2+1}}\right]^{-1}$$

$$= \log \left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$L\left(\frac{1-\cos t}{t}\right) = \log \left(\frac{\sqrt{s^2+1}}{s}\right)$$

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2) Find $L\left(\frac{e^{-3t}-e^{-4t}}{t}\right)$

Sol: $L(e^{-3t}-e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$

$$L\left(\frac{e^{-3t}-e^{-4t}}{t}\right) = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \left[\log(s+3) - \log(s+4)\right]_s^\infty$$

$$= \left(\log\left(\frac{s+3}{s+4}\right)\right)_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^{\infty}$$

$$= \log\left(\frac{s+4}{s+3}\right).$$

3) Find $L\left(\frac{1-\cos at}{t}\right)$

Sol:- $L\left(\frac{1-\cos at}{t}\right) = \int_s^{\infty} L(1-\cos at)\,ds$

$$= \int_s^{\infty}\left(\frac{1}{s} - \frac{s}{a^2+s^2}\right)ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^{\infty}$$

$$= \left(\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right)_s^{\infty}$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

$$L\left(\frac{1-\cos at}{t}\right) = \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

4) Find $L\left(\frac{\cos at - \cos bt}{t}\right)$

Sol:- $L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^{\infty} L(\cos at - \cos bt)\,ds$

$$= \int_s^\infty \left(\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2} \right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

5) Find the Laplace transform of $e^{-t}\int_0^t t\cos t\,dt$.

Sol: $L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}$

$$\left(\because L\int_0^t f(t)\,dt = \frac{1}{s}L[f(t)]\right)$$

$$= \left[\frac{1}{s}L(t\cos t)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{-d}{ds}L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

$$= \cancel{\left(s^2+1\right.} = \left[\frac{-1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}.$$

6) Evaluate using Laplace transform $\int_0^\infty t e^{-2t}\sin 3t\,dt$

Sol:- $\int_0^t t e^{-2t}\sin 3t\,dt = \int_0^\infty e^{-2t}(t\sin 3t)\,dt$

$$= \left[\int_0^\infty e^{-st}(t\sin 3t)\,dt\right]_{s=2}$$

$$= \left[L(t\sin 3t)\right]_{s=2}$$

$$= \left[\frac{-d}{ds}L(\sin 3t)\right]_{s=2}$$

$$= \left(\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right)_{s=2}$$

$$= \left(\frac{6s}{(s^2+9)^2}\right)_{s=2}$$

$$= \frac{12}{169}$$