



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

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Integral of Laplace Transform (or) Laplace transform of

$$\frac{f(t)}{t}$$

If $L[f(t)] = F(s)$ and if $\lim_{t \rightarrow 0} \frac{f(t)}{t}$ exists then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) ds$$

Proof:

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$$

Integrating w.r to 's' from s to ∞ , we get

$$\int_s^\infty F(s) ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t) dt \right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t) ds \right] dt$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st} ds \right] dt$$



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$$\begin{aligned} &= \int_0^{\infty} f(t) \left[\frac{e^{-st}}{-s} \right]_s^{\infty} dt \\ &= \int_0^{\infty} f(t) \left[0 - \frac{e^{-st}}{-s} \right] dt \\ &= \int_0^{\infty} e^{-st} \frac{f(t)}{t} dt \\ &= L \left[\frac{f(t)}{t} \right] \\ L \left[\frac{f(t)}{t} \right] &= \int_s^{\infty} F(s) ds \end{aligned}$$

Problems:-

1) Find $L \left(\frac{1-\cos t}{t} \right)$

Sol:- $L \left(\frac{1-\cos t}{t} \right) = \int_s^{\infty} L(1-\cos t) ds$

$$= \int_s^{\infty} \{ L(1) - L(\cos t) \} ds$$

$$= \int_s^{\infty} \left\{ \frac{1}{s} - \frac{s}{s^2+1} \right\} ds$$

$$= \left(\log s - \frac{1}{2} \log (s^2+1) \right)_s^{\infty}$$

$$= \left(\log s - \log (s^2+1)^{1/2} \right)_s^{\infty}$$



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$$= \left[\log \frac{s}{\sqrt{s^2+1}} \right]_s^\infty$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}} \right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1}{\sqrt{1+\frac{1}{s^2}}} \right]$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left[\frac{s}{\sqrt{s^2+1}} \right]^{-1}$$

$$= \log \left(\frac{\sqrt{s^2+1}}{s} \right)$$

$$L\left(\frac{1-\cos t}{t}\right) = \log \left(\frac{\sqrt{s^2+1}}{s} \right)$$

2) Find $L\left(\frac{e^{-3t}-e^{-4t}}{t}\right)$

$$\text{Sol: } L(e^{-3t}-e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$$

$$L\left(\frac{e^{-3t}-e^{-4t}}{t}\right) = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4} \right) ds$$

$$= \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4} \right) ds$$

$$= \left[\log(s+3) - \log(s+4) \right]_s^\infty$$

$$= \left(\log \left(\frac{s+3}{s+4} \right) \right)_s^\infty$$



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$$= \left[\log \left(\frac{s+3}{s+4} \right) \right]_s^{\infty}$$
$$= \log \left(\frac{s+4}{s+3} \right).$$

3) Find $L \left(\frac{1 - \cos at}{t} \right)$

Sol. $L \left(\frac{1 - \cos at}{t} \right) = \int_s^{\infty} L(1 - \cos at) ds$

$$= \int_s^{\infty} \left(\frac{1}{s} - \frac{s}{a^2 + s^2} \right) ds$$

$$= \left(\log s - \frac{1}{2} \log(s^2 + a^2) \right) \Big|_s^{\infty}$$

$$= \left(\log \left(\frac{s}{\sqrt{s^2 + a^2}} \right) \right) \Big|_s^{\infty}$$

$$= 0 - \log \left(\frac{s}{\sqrt{s^2 + a^2}} \right)$$

$$= \log \left(\frac{\sqrt{s^2 + a^2}}{s} \right)$$

$$L \left(\frac{1 - \cos at}{t} \right) = \log \left(\frac{\sqrt{s^2 + a^2}}{s} \right)$$

4) Find $L \left(\frac{\cos at - \cos bt}{t} \right)$

Sol. $L \left(\frac{\cos at - \cos bt}{t} \right) = \int_s^{\infty} L(\cos at - \cos bt) ds$



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$$\begin{aligned}
&= \int_s^\infty \left(\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2} \right) ds \\
&= \frac{1}{2} \left[\log(s^2+a^2) - \log(s^2+b^2) \right]_s^\infty \\
&= \frac{1}{2} \left[\log \frac{s^2+a^2}{s^2+b^2} \right]_s^\infty \\
&= \frac{1}{2} \left[0 - \log \frac{s^2+a^2}{s^2+b^2} \right] \\
&= -\frac{1}{2} \log \left(\frac{s^2+a^2}{s^2+b^2} \right)
\end{aligned}$$

5) Find the Laplace transform of $e^{-t} \int_0^t t \cos t dt$

Sol: $L \left[e^{-t} \int_0^t t \cos t dt \right] = \left[L \left(\int_0^t t \cos t dt \right) \right]_{s \rightarrow s+1}$

$\left(\because L \int_0^t f(t) dt = \frac{1}{s} L[f(t)] \right)$

$$\begin{aligned}
&= \left[\frac{1}{s} L(t \cos t) \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} \left(-\frac{d}{ds} L(\cos t) \right) \right]_{s \rightarrow s+1} \\
&= \left[-\frac{1}{s} \frac{d}{ds} \left(\frac{s}{s^2+1} \right) \right]_{s \rightarrow s+1} \\
&= \left[-\frac{1}{s} \left(\frac{s^2+1-2s^2}{(s^2+1)^2} \right) \right]_{s \rightarrow s+1} \\
&= \left[-\frac{1}{s} \left(\frac{1-s^2}{(s^2+1)^2} \right) \right]_{s \rightarrow s+1}
\end{aligned}$$



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$$= \left[\frac{s^2 - 1}{s(s^2 + 1)^2} \right]_{s \rightarrow s+1}$$

$$= \frac{(s+1)^2 - 1}{(s+1)(s^2 + 1)^2}$$

$$= \frac{s^2 + 2s}{(s+1)(s^2 + 2s + 2)^2}$$

$$L \left[e^{-t} \int_0^t t \cos t \, dt \right] = \frac{s^2 + 2s}{(s+1)(s^2 + 2s + 2)^2}$$

6) Evaluate using Laplace transform $\int_0^\infty t e^{-2t} \sin 3t \, dt$

$$\text{sol: } \int_0^\infty t e^{-2t} \sin 3t \, dt = \int_0^\infty e^{-2t} (t \sin 3t) \, dt$$

$$= \left[\int_0^\infty e^{-st} (t \sin 3t) \, dt \right]_{s=2}$$

$$= [L(t \sin 3t)]_{s=2}$$

$$= \left[-\frac{d}{ds} L(\sin 3t) \right]_{s=2}$$

$$= \left(-\frac{d}{ds} \left(\frac{3}{s^2 + 9} \right) \right)_{s=2}$$

$$= \left(\frac{6s}{(s^2 + 9)^2} \right)_{s=2}$$

$$= \frac{12}{169}$$