Properties:

Change of Scale Property:

If $L\{f(t)\} = F(s)$, then

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Proof:

We know that,

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$$

Put $at = x$

$$t = \frac{x}{a} \qquad dt = \frac{dx}{a}$$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)x} f(x)\, dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} f(t)\, dt$$

$$\boxed{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

First Shifting Property:

If $L\{f(t)\} = F(s)$ then

i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof:

i) We know that,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st}[e^{-at} f(t)]dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

ii) $$L(e^{at} f(t)) = \int_0^\infty e^{-st}[e^{at} f(t)]dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

Second shifting Property:

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$.

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st}.g(t)\,dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a=u \Rightarrow dt = du$

when $t=a \Rightarrow u=0$

$t \to \infty \Rightarrow u=\infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} \cdot e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\,dt \quad \text{Replace } u \to t$$

$$= e^{-as} F(s)$$

$$\boxed{L[g(t)] = e^{-as} F(s)}$$

Problems on Change of Scale Property:

1) Find $L(\sinh 3t)$ by using change of scale Property.

Sol: $L(\sinh t) = \dfrac{1}{s^2-1} = F(s)$

$$L(\sinh 3t) = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \int_a^{\infty} e^{-st} f(t-a)\,dt$$

Put $t-a=u \Rightarrow dt = du$

when $t = a \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

$$L[g(t)] = \int_0^{\infty} e^{-s(u+a)} f(u)\,du$$

$$= \int_0^{\infty} e^{-us} \cdot e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^{\infty} e^{-us} f(u)\,du$$

$$= e^{-as} \int_0^{\infty} e^{-st} f(t)\,dt \qquad \text{Replace } u \to t$$

$$= e^{-as} F(s)$$

$$\boxed{L[g(t)] = e^{-as} F(s)}$$

Problems on Change of Scale Property:

1) Find $L(\sinh 3t)$ by using change of scale Property.

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$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3}\left(\frac{9}{s^2-9}\right)$$

$$= \frac{3}{s^2-9}$$

$$L(\sinh 3t) = \frac{3}{s^2-9}$$

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2) Find $L(\cos 5t)$ using change of scale Property.

Sol: $L(\cos t) = \frac{s}{s^2+1} = F(s)$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{\left(\frac{s}{5}\right)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$L(\cos 5t) = \frac{s}{s^2+25}$$

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3) Given $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$ applying the change of Scale Property show that $L[f(2t)] = \frac{s^2-2s+4}{4(s+1)^2(s-2)}$.

Sol: $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$

$$L[f(2t)] = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\frac{(s/2)^2 - (s/2) + 1}{(2s/2+1)^2\left(\frac{s}{2}-1\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2-2s+4}{4}}{\frac{4}{(s+1)^2(s-2)/2}}\right]$$

$$= \frac{1}{4}\,\frac{s^2-2s+4}{(s+1)^2(s-2)}$$

$$L[f(2t)] = \frac{s^2-2s+4}{4(s+1)^2(s-2)}$$

4) Find $L(e^{5t})$ applying change of scale property.

Sol: $L(e^{t}) = \frac{1}{s-1} = F(s)$

$$L(e^{5t}) = \frac{1}{5}F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\,\frac{1}{\frac{s}{5}-1}$$

$$= \frac{1}{5}\,\frac{5}{s-5}$$

$$= \frac{1}{s-5}$$

$$L(e^{5t}) = \frac{1}{s-5}$$

Problems on first shifting theorem:

1) Find $L(e^{-3t}\sin^2 t)$

Sol: $L(e^{-at}f(t)) = F(s+a)$

$$L(e^{-3t}\sin^2 t) = L(\sin^2 t)_{s\to s+3}$$

$$= L\left(\frac{1-\cos 2t}{2}\right)_{s\to s+3}$$

$$= \frac{1}{2}\left\{L(1) - L(\cos 2t)\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{(s+3)((s+3)^2+4)}\right\}$$

$$= \frac{2}{(s+3)[(s+3)^2+4]}$$

$$L(e^{-3t}\sin^2 t) = \frac{2}{(s+3)[(s+3)^2+4]}$$

2) Find $L(t^2 e^{-2t})$.

Sol: $L(e^{-at}f(t)) = F(s+a)$

$$L(e^{-2t}t^2) = \left[L(t^2)\right]_{s\to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s\to s+2}$$

$$= \left(\frac{2}{s^3}\right)_{s \to s+2}$$

$$= \frac{2}{(s+2)^3}$$

$$L(t^2 e^{-2t}) = \frac{2}{(s+2)^3}$$

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3) Find $L[e^{2t} \cos 5t]$

Sol:. $L[e^{2t}\cos 5t] = L[\cos 5t]_{s \to s-2}$

$$= \left[\frac{s}{s^2+25}\right]_{s \to s-2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

$$L[e^{2t}\cos 5t] = \frac{s-2}{(s-2)^2+25}$$

Problems on Second Shifting theorem:

1) Find $L[f(t)]$ where $f(t) = \begin{cases} 0 & , 0 < t < 2 \\ 3 & , t > 2 \end{cases}$

Sol:. $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= 0 + \int_2^\infty e^{-st}\cdot 3\, dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^{\infty}$$

$$= \frac{-3}{s}\left(e^{-\infty} - e^{-2s}\right)$$

$$= \frac{3e^{-2s}}{s}$$

$$L[f(t)] = \frac{3e^{-2s}}{s}$$

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2) Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi. \end{cases}$$

Sol:

$$f(t)\begin{cases} = \sin t, & 0<t<\pi \\ = 0, & \pi<t<2\pi \end{cases}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{\pi} e^{-st} f(t)\,dt + \int_{\pi}^{\infty} e^{-st}.0\,dt$$

$\left\{ \because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \right.$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^{\pi}$$

$$= \frac{e^{-s\pi}}{s^2+1}(-s\sin\pi - \cos\pi) + \frac{1}{s^2+1}$$

$$= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

$$L[f(t)] = \frac{1+e^{-\pi s}}{s^2+1}$$