Applications of Laplace transforms to Differential Equations:

If $L[f(t)] = F(s)$ then

$L[y'(t)] = sL(y) - y(0)$

$L[y''(t)] = s^2L(y) - sy(0) - y'(0)$.

1) Solve the differential equations using Laplace transforms $y''+4y'+4y = e^{-t}$ given that $y(0)=0$ and $y'(0)=0$.

Sol: $y''+4y'+4y = e^{-t}$

Taking Laplace transforms we get,

$$L(y''+4y'+4y) = L(e^{-t})$$

$$L(y'') + 4L(y') + 4L(y) = \frac{1}{s+1}$$

$$[s^2L(y) - sy(0) - y'(0)] + 4[sL(y) - y(0)] + 4L(y) = \frac{1}{s+1}$$

Given: $y(0) = 0$, $y'(0) = 0$

$$[s^2L(y) - s\times 0 - 0] + 4[sL(y) - 0] + 4L(y) = \frac{1}{s+1}$$

$$s^2L(y) + 4sL(y) + 4L(y) = \frac{1}{s+1}$$

$$(s^2+4s+4)L(y) = \frac{1}{s+1}$$

$$L(y)(s+2)^2 = \frac{1}{s+1}$$

$$L(y) = \frac{1}{(s+1)(s+2)^2}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

$s = -2 \Rightarrow C = -1$

$s = -1 \Rightarrow A = 1$

$s = 0 \Rightarrow B = -1$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= e^{-t} - e^{-2t} - te^{-2t}$$

$$L^{-1} = \frac{1}{(s+2)^2}$$

$$= e^{-2t} \cdot L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-2t} \cdot t.$$