Properties:

Change of Scale Property:

If $L\{f(t)\} = F(s)$, then

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Proof:

We know that,

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$$

Put $at = x$

$$t = \frac{x}{a} \qquad dt = \frac{dx}{a}$$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x)\,\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)x} f(x)\,dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} f(t)\,dt$$

$$\boxed{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

First Shifting Property:

If $L\{f(t)\} = F(s)$ then

i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof:

i) We know that,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st}[e^{-at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

ii) $$L(e^{at} f(t)) = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

Second shifting Property:

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$.

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a=u \Rightarrow dt = du$

when $t=a \Rightarrow u=0$

$t=\infty \Rightarrow u=\infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} \cdot e^{-as} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt \qquad \text{Replace } u \to t$$

$$= e^{-as} F(s)$$

$$\boxed{L[g(t)] = e^{-as} F(s)}$$

Problems on Change of Scale Property:.

1) Find $L(\sinh 3t)$ by using change of scale Property.

Sol: $L(\sinh t) = \dfrac{1}{s^2-1} = F(s)$

$$L(\sinh 3t) = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2-1}$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2-1}$$

$$= \frac{1}{3}\left(\frac{9}{s^2-9}\right)$$

$$= \frac{3}{s^2-9}$$

$$L(\sinh 3t) = \frac{3}{s^2-9}$$

2) Find $L(\cos 5t)$ using change of scale Property.

Sol: $L(\cos t) = \frac{s}{s^2+1} = F(s)$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{\left(\frac{s}{5}\right)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$L(\cos 5t) = \frac{s}{s^2+25}$$

3) Given $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$ applying the change of Scale Property show that $L[f(2t)] = \frac{s^2-2s+4}{4(s+1)^2(s-2)}$.

Sol: $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$

$$L[f(2t)] = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\frac{(s/2)^2-(s/2)+1}{(2s/2+1)^2\left(\frac{s}{2}-1\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2-2s+4}{4}}{\frac{4}{(s+1)^2(s-2)/2}}\right]$$

$$= \frac{1}{4}\,\frac{s^2-2s+4}{(s+1)^2(s-2)}$$

$$\boxed{L[f(2t)] = \frac{s^2-2s+4}{4(s+1)^2(s-2)}}$$

4) Find $L(e^{5t})$ applying change of scale property.

Sol: $L(e^{t}) = \frac{1}{s-1} = F(s)$

$$L(e^{5t}) = \frac{1}{5}F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\,\frac{1}{\frac{s}{5}-1}$$

$$= \frac{1}{\cancel{5}}\,\frac{\cancel{5}}{s-5}$$

$$= \frac{1}{s-5}$$

$$L(e^{5t}) = \frac{1}{s-5}$$