



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



Laplace transforms of derivatives ::

If $L[f(t)] = F(s)$ then $L[f'(t)] = sF(s) - f(0)$.

Proof ::

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

Integrating by Parts, we get

$$= [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} f(t) (-se^{-st}) dt$$

$$= [e^{-\infty} f(\infty) - e^0 f(0)] + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= sF(s) - f(0)$$

$$\boxed{L[f'(t)] = sF(s) - f(0)}$$

Corollary ::

$$\text{Let } f''(t) = s^2 F(s) - sf(0) - f'(0)$$

$$\text{Let } L[g'(t)] = sG(s) - g(0)$$

we know that,

$$L[f'(t)] = sL[f(t)] - f(0)$$

Replace $f(t) \rightarrow f'(t)$ & $f'(t) \rightarrow f''(t)$ & $f(0) \rightarrow f'(0)$

$$L[f''(t)] = sL[f'(t)] - f'(0)$$

$$= s[sL[f(t)] - f(0)] - f'(0)$$

$$= s^2 L[f(t)] - sf(0) - f'(0)$$

$$= s^2 F(s) - sf(0) - f'(0)$$



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Laplace transform of integrals:

$$\text{If } L[f(t)] = F(s) \text{ then } L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Proof:

$$\text{Let } g(t) = \int_0^t f(t) dt \text{ and } g(0) = 0 \text{ then } g'(t) = f(t)$$

We know that

$$\begin{aligned} L[g'(t)] &= sL(g(t) - g(0)) \\ &= sL(g(t)) \end{aligned}$$

$$L[g(t)] = \frac{1}{s} L[g'(t)]$$

$$L\left[\int_0^t f(t) dt\right] = \frac{1}{s} L[f(t)]$$

$$L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Derivative of Laplace Transform (or) Laplace transform of

$t f(t)$

$$\text{If } L[f(t)] = F(s) \text{ then}$$

$$L[tf(t)] = -\frac{d}{ds} F(s)$$

Proof:

We know that,

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt$$



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$$= \int_0^{\infty} \frac{\partial}{\partial s} (e^{-st}) f(t) dt$$

$$= \int_0^{\infty} -t e^{-st} f(t) dt$$

$$= - \int_0^{\infty} e^{-st} t f(t) dt$$

$$= -L[t f(t)]$$

$$L[t f(t)] = -\frac{d}{ds} [F(s)].$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)].$$

Laplace Transforms of Derivatives:

1) Find $L[t \sin at]$

Sol: $f(t) = t \sin at$

$$f'(t) = a t \cos at + \sin at$$

$$f''(t) = a [-a t \sin at + \cos at] + a \cos at$$

$$= 2a \cos at - a^2 t \sin at$$

$$f(0) = 0, f'(0) = 0$$

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

$$L[2a \cos at - a^2 t \sin at] = s^2 L[t \sin at] - s(0) - 0$$



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$$2aL(\cos at) - a^2L(t\sin at) = s^2L(t\sin at)$$

$$(s^2 + a^2)L(t\sin at) = 2aL(\cos at)$$

$$(s^2 + a^2)L(t\sin at) = 2a \cdot \frac{s}{a^2 + s^2}$$

$$L(t\sin at) = \frac{2as}{(s^2 + a^2)^2}$$

1) Find $L[t\cos at]$

$$\underline{\text{Sol:}} \quad L[tf(t)] = -\frac{d}{ds} [L(f(t))]$$

$$L[t\cos at] = -\frac{d}{ds} [L(\cos at)]$$

$$= -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right)$$

$$= -\left\{ \frac{s^2 + a^2 - s(2s)}{(s^2 + a^2)^2} \right\}$$

$$= -\left\{ \frac{a^2 - s^2}{(s^2 + a^2)^2} \right\}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$L[t\cos at] = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

2) Find $L[te^{2t} \sin 3t]$

$$\underline{\text{Sol:}} \quad L[te^{2t} \sin 3t] = -\frac{d}{ds} \{L(e^{2t} \sin 3t)\}$$



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$$= -\frac{d}{ds} \left\{ L(\sin 3t) \right\} s \rightarrow s-2$$

$$= -\frac{d}{ds} \left\{ \left(\frac{3}{s^2+9} \right) s \rightarrow s-2 \right\}$$

$$= - \left\{ \frac{-3(2s)}{(s^2+9)^2} \right\} s \rightarrow s-2$$

$$= \left\{ \frac{6s}{(s^2+9)^2} \right\} s \rightarrow s-2$$

$$= \frac{6(s-2)}{((s-2)^2+9)^2}$$

$$L[e^{2t} \sin 3t] = \frac{6(s-2)}{(s^2-4s+13)^2}$$

3) Find $L[t^2 e^{-2t} \cos t]$

$$\text{Sol: } L[t^2 e^{-2t} \cos t] = (-1)^2 \frac{d^2}{ds^2} \left\{ L(e^{-2t} \cos t) \right\}$$

$$= \frac{d^2}{ds^2} \left\{ L(\cos t) s \rightarrow s+2 \right\}$$

$$= \frac{d^2}{ds^2} \left\{ \frac{s}{s^2+1} \right\} s \rightarrow s+2$$

$$= \frac{d}{ds} \left\{ \frac{s^2+1 - s(2s)}{(s^2+1)^2} \right\} s \rightarrow s+2$$

$$= \frac{d}{ds} \left\{ \frac{1-s^2}{(s^2+1)^2} \right\} s \rightarrow s+2$$



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$$= \frac{d}{ds} \left\{ \frac{1-s^2}{(s^2+1)^2} \right\}_{s \rightarrow s+2}$$

$$= \left\{ \frac{(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)(2s)}{(s^2+1)^3} \right\}_{s \rightarrow s+2}$$

$$= \left\{ \frac{(s^2+1)(-2s) - 4s(1-s^2)}{(s^2+1)^3} \right\}_{s \rightarrow s+2}$$

$$= \frac{[(s+2)^2+1] [-2(s+2)] - 4(s+2)[1-(s+2)^2]}{((s+2)^2+1)^3}$$

$$= \frac{(s^2+4s+5)(-2s-4) + (4s+8)(s^2+4s+3)}{((s+2)^2+1)^3}$$

$$= \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

$$L[t^2 e^{-2t} \cos t] = \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

4) Find $L\left[\frac{\sin 3t}{t}\right]$

Sol: $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) ds = \int_s^\infty L[f(t)] ds$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L(\sin 3t) ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right) ds$$



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$$\begin{aligned} &= \int_s \frac{3}{s^2 + 3^2} ds \\ &= 3 \cdot \frac{1}{3} \left[\tan^{-1} \left(\frac{s}{3} \right) \right]_s^\infty \\ &= \tan^{-1}(\infty) - \tan^{-1}(s/3) \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) \\ &= \cot^{-1}\left(\frac{s}{3}\right) \end{aligned}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right).$$