$$= \int_s^{\infty} \frac{3}{s^2+3^2}\, ds$$

$$= 3 \cdot \frac{1}{3} \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^{\infty}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \cot^{-1}(s/3)$$

Integral of Laplace Transform (or) Laplace transform of $\frac{f(t)}{t}$:

If $L[f(t)] = F(s)$ and if $\lim_{t\to 0} \frac{f(t)}{t}$ exists then

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\,ds$$

Proof:

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

Integrating w.r.to 's' from s to ∞, we get

$$\int_s^{\infty} F(s)\,ds = \int_s^{\infty} \left[\int_0^{\infty} e^{-st} f(t)\,dt\right] ds.$$

$$= \int_0^{\infty} \left[\int_s^{\infty} e^{-st} f(t)\,ds\right] dt$$

$$= \int_0^{\infty} f(t) \left[\int_s^{\infty} e^{-st}\,ds\right] dt$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-s}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$$

Problems:-

1) Find $L\left(\frac{1-\cos t}{t}\right)$

Sol:- $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty L(1-\cos t)\, ds$

$$= \int_s^\infty \{L(1) - L(\cos t)\} ds$$

$$= \int_s^\infty \left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+1)\right)_s^\infty$$

$$= \left(\log s - \log(s^2+1)^{1/2}\right)_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log \left[\frac{s}{\sqrt{s^2+1}}\right]^{-1}$$

$$= \log \left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$L\left(\frac{1-\cos t}{t}\right) = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

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2) Find $L\left(\frac{e^{-3t} - e^{-4t}}{t}\right)$

Sol: $L(e^{-3t} - e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$

$$L\left(\frac{e^{-3t} - e^{-4t}}{t}\right) = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \left[\log(s+3) - \log(s+4)\right]_s^\infty$$

$$= \left(\log\left(\frac{s+3}{s+4}\right)\right)_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^\infty$$

$$= \log\left(\frac{s+4}{s+3}\right).$$

3) Find $L\left(\frac{1-\cos at}{t}\right)$

Sol: $L\left(\frac{1-\cos at}{t}\right) = \int_s^\infty L(1-\cos at)\,ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{a^2+s^2}\right) ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left(\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right)_s^\infty$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

$$L\left(\frac{1-\cos at}{t}\right) = \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

4) Find $L\left(\frac{\cos at - \cos bt}{t}\right)$

Sol: $L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty L(\cos at - \cos bt)\,ds$

$$= \int_s^\infty \left(\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2} \right) ds$$

$$= \frac{1}{2} \left[\log(s^2+a^2) - \log(s^2+b^2) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log \frac{s^2+a^2}{s^2+b^2} \right]_s^\infty$$

$$= \frac{1}{2} \left[0 - \log \frac{s^2+a^2}{s^2+b^2} \right]$$

$$= \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2} \right)$$

5) Find the Laplace transform of $e^{-t} \int_0^t t\cos t\, dt$.

Sol: $L\left[e^{-t} \int_0^t t\cos t\, dt\right] = \left[L\left(\int_0^t t\cos t\, dt \right) \right]_{s \to s+1}$

$$\left(\because L \int_0^t f(t)\, dt = \frac{1}{s} L[f(t)] \right)$$

$$= \left[\frac{1}{s} L(t\cos t) \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \left(\frac{-d}{ds} L(\cos t) \right) \right]_{s \to s+1}$$

$$= \left[\frac{-1}{s} \frac{d}{ds} \left(\frac{s}{s^2+1} \right) \right]_{s \to s+1}$$

$$= \left[\frac{-1}{s} \left(\frac{s^2+1-2s^2}{(s^2+1)^2} \right) \right]_{s \to s+1}$$

$$= \left[\frac{-1}{s} \left(\frac{1-s^2}{(s^2+1)^2} \right) \right]_{s \to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

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6) Evaluate using Laplace transform $\int_0^\infty t e^{-2t}\sin 3t\,dt$

Sol:- $$\int_0^t t e^{-2t}\sin 3t\,dt = \int_0^\infty e^{-2t}(t\sin 3t)\,dt$$

$$= \left[\int_0^\infty e^{-st}(t\sin 3t)\,dt\right]_{s=2}$$

$$= \left[L(t\sin 3t)\right]_{s=2}$$

$$= \left[\frac{-d}{ds}L(\sin 3t)\right]_{s=2}$$

$$= \left(\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right)_{s=2}$$

$$= \left(\frac{6s}{(s^2+9)^2}\right)_{s=2}$$

$$= \frac{12}{169}$$