



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



Initial value theorem:

If the Laplace transform of $f(t)$ and $f'(t)$ exists and

$L[f(t)] = F(s)$ then

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Proof:

We know that

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$= sF(s) - f(0)$$

$$sF(s) = L[f'(t)] + f(0)$$

$$sF(s) = \int_0^{\infty} e^{-st} f'(t) dt + f(0)$$

Taking limit as $s \rightarrow \infty$ on both sides, we get

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \left\{ \int_0^{\infty} e^{-st} f'(t) dt + f(0) \right\}$$

$$= \lim_{s \rightarrow \infty} \int_0^{\infty} e^{-st} f'(t) dt + f(0)$$

$$= \int_0^{\infty} \lim_{s \rightarrow \infty} e^{-st} f'(t) dt + f(0)$$

$$= 0 + f(0)$$

$$= \lim_{t \rightarrow 0} f(t)$$

$$\text{Hence, } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$



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Problems:-

1) Verify the initial and final value theorem for

$$f(t) = 1 + e^{-t}(\sin t + \cos t)$$

Sol:- $F(s) = L[1 + e^{-t}\sin t + e^{-t}\cos t]$

$$= L(1) + L(\sin t)_{s \rightarrow s+1} + L(\cos t)_{s \rightarrow s+1}$$

$$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \rightarrow s+1} + \left(\frac{s}{s^2+1}\right)_{s \rightarrow s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$SF(s) = s \left[\frac{1}{s} + \frac{s+2}{s^2+2s+2} \right]$$

Initial value theorem $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} s F(s)$

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} [1 + e^{-t}(\sin t + \cos t)] = 1 + 1 = 2$$

$$\lim_{s \rightarrow \infty} s F(s) = \lim_{s \rightarrow \infty} s \left(\frac{1}{s} + \frac{s+2}{s^2+2s+2} \right)$$

$$= \lim_{s \rightarrow \infty} \left(1 + \frac{s^2+2s}{s^2+2s+2} \right)$$

$$= 1 + 1$$

$$= 2$$

$$\text{Hence } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} s F(s) = 2$$

Initial value theorem is Verified.



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Final Value theorem: $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} \int_0^t [1 + e^{-t}(\sin t + \cos t)] dt = 1$$

$$\lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} s \left[\frac{1}{s} + \frac{s+2}{s^2+2s+2} \right]$$

$$= 1$$

$$\text{Hence } \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) = 1$$

$\therefore$ Final value theorem is verified.

Laplace transform of some Special functions:

Unit step function:

The unit step function also called Heavisides unit function

is defined as,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$.

Result:

Laplace Transform of unit step function is $\frac{e^{-as}}{s}$.

$$\text{i.e., } L[u(t-a)] = \frac{e^{-as}}{s}$$



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Proof:-

$$\begin{aligned} L[u(t-a)] &= \int_0^{\infty} e^{-st} u(t-a) dt \\ &= \int_a^{\infty} e^{-st} u(t-a) dt + \int_0^a e^{-st} u(t-a) dt \\ &= 0 + \int_a^{\infty} e^{-st} dt \\ &= \left[\frac{e^{-st}}{-s} \right]_a^{\infty} \end{aligned}$$

$$L[u(t-a)] = \frac{e^{-as}}{s} \quad (s > 0)$$

Transforms of Periodic functions:

A function $f(x)$ is said to be Periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x . The Smallest Positive value of p for which this equation is true for every value of x will be called the Period of the function.

The Laplace transformation of a periodic function $f(t)$ with Period p given by,

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt$$



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Problems:

1) Find the Laplace transform of the rectangular wave given by $f(t) = \begin{cases} 1, & 0 \leq t < b \\ -1, & b \leq t < 2b \end{cases}$

Sol: Given: $f(t) = \begin{cases} 1, & 0 \leq t < b \\ -1, & b \leq t < 2b \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt$$

This function is Periodic in the interval $(0, 2b)$ with Period

$2b$.

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} dt + \int_b^{2b} e^{-st} (-1) dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\left(\frac{e^{-st}}{-s} \right)_0^b - \left(\frac{e^{-st}}{-s} \right)_b^{2b} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[-\frac{1}{s} (e^{-st})_0^b + \frac{1}{s} (e^{-st})_b^{2b} \right]$$

$$= \frac{1}{s(1-e^{-2bs})} \left[-(e^{-bs} - 1) + (e^{-2bs} - e^{-bs}) \right]$$

$$= \frac{-e^{-bs} + 1 + (e^{-bs})^2 - e^{-bs}}{s(1-e^{-2bs})}$$



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$$\begin{aligned}
&= \frac{1 - 2e^{-bs} + e^{-2bs}}{s(1 - e^{-2bs})} \\
&= \frac{1}{s(1 - e^{-bs})(1 + e^{-bs})} (1 - e^{-bs})^2 \\
&= \frac{1}{s} \left(\frac{1 - e^{-bs}}{1 + e^{-bs}} \right) \\
&= \frac{1}{s} \left(\frac{e^{sb/2} - e^{-sb/2}}{e^{sb/2} + e^{-sb/2}} \right) \\
&= \frac{1}{s} \tanh\left(\frac{bs}{2}\right).
\end{aligned}$$

2) Find the Laplace transform of the half wave rectifies

function $f(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$

Sol: $L[f(t)] = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t dt + 0 \right]$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1 - e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega} \cdot \omega + \omega}{s^2 + \omega^2} \right]$$



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$$= \frac{\omega (1 + e^{-\pi/\omega})}{(1 - e^{-\pi/\omega}) (1 + e^{-\pi/\omega}) (s^2 + \omega^2)}$$

$$= \frac{\omega}{(1 - e^{-\pi/\omega}) (s^2 + \omega^2)}$$

3) Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases} \quad \text{with } f(t + 2a) = f(t).$$

Q0:- $L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt.$

$$= \frac{1}{1 - e^{-2as}} \left[\int_0^a e^{-st} t dt + \int_a^{2a} e^{-st} (2a - t) dt \right]$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \left[t \left(\frac{e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{s^2} \right) \right]_0^a + \left[(2a - t) \left(\frac{e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{s^2} \right) \right]_a^{2a} \right\}$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \left[-t \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a - t) \frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \left[\left(-a \frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(-\frac{1}{s^2} \right) \right] + \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{1 - e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$



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$$\begin{aligned} &= \frac{1}{1-e^{-2as}} \left[\frac{1+e^{-2as}-2e^{-as}}{s^2} \right] \\ &= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})} \\ &= \frac{1-e^{-as}}{s^2(1+e^{-as})} \\ &= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right) \end{aligned}$$