Convolution :-

Definition :-

If $f(t)$ and $g(t)$ are two functions defined for $t \geq 0$ then the Convolution of $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note :- $f(t) * g(t) = g(t) * f(t)$.

Convolution theorem :-

If $f(t)$ and $g(t)$ are two laplace transformable functions defined for $t \geq 0$ then $L[f(t) * g(t)]$ is given by

$$L^{-1}[F(s).G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems:

1) Using Convolution theorem, find the inverse transform of

i) $\frac{s^2}{(s^2+a^2)^2}$ ii) $\frac{s}{(s^2+a^2)(s^2+b^2)}$ iii) $\frac{1}{(s+a)(s+b)}$.

Sol: i) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \cos a(t-u)\,du$$

$$= \int_0^t \left[\frac{\cos(\cancel{au}+at-\cancel{au})+\cos(au-at+au)}{2}\right]du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au-at)]\,du$$

$$= \frac{1}{2}\left[\cos at.u + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[\cos at.t + \frac{\sin(2at-at)}{2a} - \frac{\sin(0-at)}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a}(at\cos at + \sin at)$$

ii) $L^{-1}\left[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right]$

$$L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \cos at * \frac{1}{b}L^{-1}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{1}{b}\int_0^t \cos au \sin b(t-u)\,du$$

$$= \frac{1}{b}\int_0^t \frac{\sin(bt-bu+au)+\sin(bt-bu-au)}{2}\,du$$

$$= \frac{1}{2b}\int_0^t \sin[bt+(a-b)u] + \sin(bt-(a+b)u)\,du$$

$$= \frac{1}{2b}\left\{\frac{-\cos(bt+(a-b)u}{a-b} - \frac{\cos(bt-(a+b)u}{-(a+b)}\right\}_0^t$$

$$= \frac{1}{2b}\left\{\frac{-\cos(\cancel{bt}+at-\cancel{bt})}{a-b} + \frac{\cos bt}{a-b} + \frac{\cos(\cancel{bt}-at-\cancel{bt})}{a+b} - \frac{\cos bt}{a+b}\right\}$$

$$= \frac{1}{2b}\left\{\frac{-\cos at}{a-b} + \frac{\cos bt}{a-b} + \frac{\cos at}{a+b} - \frac{\cos bt}{a+b}\right\}$$

$$= \frac{1}{2b}\left\{-\frac{\cos at}{a-b} + \frac{\cos bt}{a-b} + \frac{\cos at}{a+b} - \frac{\cos bt}{a+b}\right\}$$

$$= \frac{1}{2b}\left\{\frac{-1}{a-b}[\cos at - \cos bt] + \frac{1}{a+b}(\cos at - \cos bt)\right\}$$

$$= \frac{1}{2b}(\cos at - \cos bt)\left\{-\frac{1}{a-b} + \frac{1}{a+b}\right\}$$

$$= \frac{1}{2b}\{\cos at - \cos bt\}\left(\frac{-(a+b)+(a-b)}{a^2-b^2}\right)$$

$$= \frac{1}{2b}(\cos at - \cos bt)\frac{(-2b)}{a^2-b^2}$$

$$= \frac{\cos bt - \cos at}{a^2-b^2}$$

iii) $$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left(\frac{1}{s+a}\cdot\frac{1}{s+b}\right)$$

$$= L^{-1}\left(\frac{1}{s+a}\right) * L^{-1}\left(\frac{1}{s+b}\right)$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}\cdot e^{-b(t-u)}\,du$$

$$= \int_0^t e^{-au-bt+bu}\,du$$

$$= \left[\frac{e^{-au-bt+bu}}{-a+b}\right]_0^t$$

$$= \left[\frac{e^{-at-bt+bt}}{-a+b} - \frac{e^{-bt}}{-a+b}\right]$$

$$= \frac{e^{-at}-e^{-bt}}{b-a}$$