ODD AND EVEN FUNCTIONS:

EVEN FUNCTION:

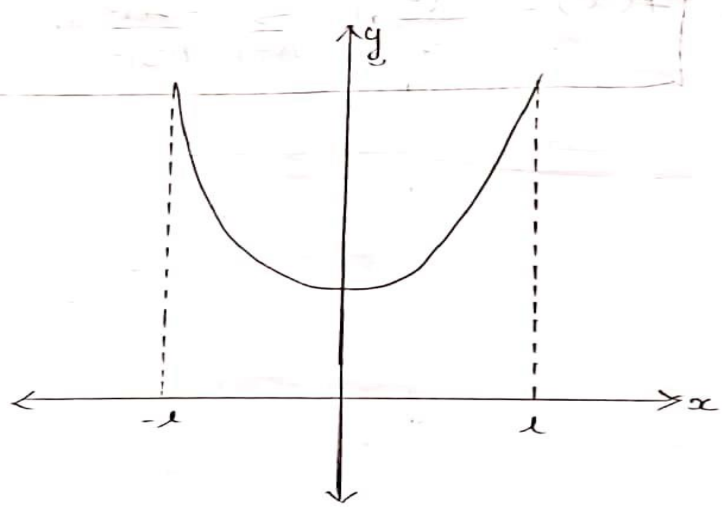
Let $f(x)$ be defined in $(-l, l)$

If $f(-x) = f(x)$, then $f(x)$ is an even function.

Note:

1. The graph of even function is symmetrical about y-axis
2. $\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$ if $f(x)$ is even.
3. Sum of two even functions is also an even function.
4. Product of two even functions is also an even function.

Graph of an Even Function:



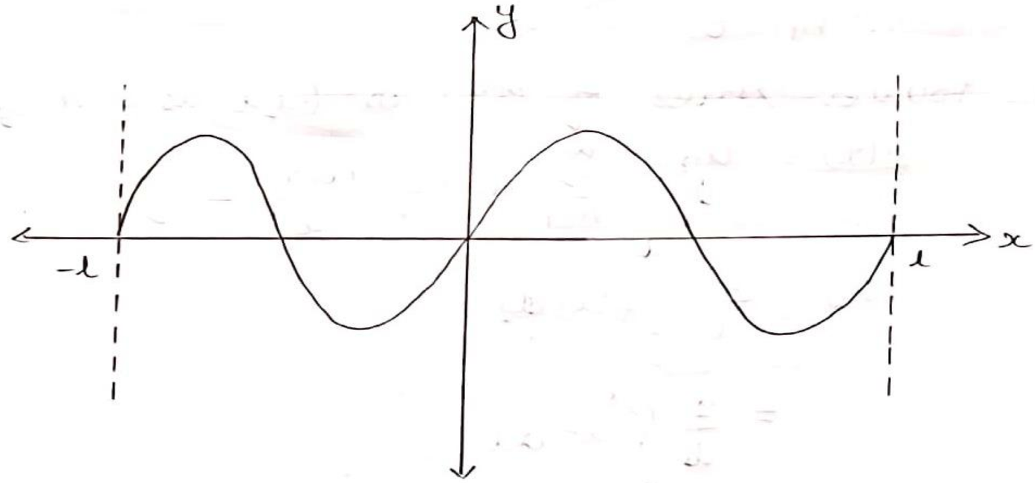
ODD FUNCTION:

Let $f(x)$ be defined in $(-l, l)$.
If $f(-x) = -f(x)$, then $f(x)$ is an odd function

Note:

1. The graph of odd function is symmetrical about origin.
2. $\int_{-l}^{l} f(x)\,dx = 0$ if $f(x)$ is odd
3. Sum of two odd function is also an odd function.
4. Product of two odd function is an even function.

Graph of an Odd Function:



PROBLEMS UNDER ODD AND EVEN FUNCTIONS:

1) Find the Fourier series for $f(x) = x^2$ in $(-l, l)$ and hence deduce

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$.

Solution:

$f(x)$ is defined in $(-l, l)$

In $(-l, l)$, check whether $f(x)$ is even / odd.

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$$f(x) = f(-x)$$

Here, $f(x)$ is an even function.

$$\therefore b_n = 0.$$

The Fourier series of $f(x)$ in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{—} (1)$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{l} \int_0^l x^2\, dx$$

$$= \frac{2}{l} \left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left(\frac{l^3}{3}\right)$$

$$\boxed{a_0 = \frac{2l^2}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (2)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi x}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{4}{l}\left(\frac{l}{n\pi}\right)^2\left[x \cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l}{n^2\pi^2}\left[l \cos n\pi - 0 \cos 0\right]$$

$$\boxed{a_n = \frac{4l^2(-1)^n}{n^2\pi^2}}$$

Substitute a_0, a_n in (1)

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{l} \quad \text{— (2)}$$

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

Put $x = l$ in (2), we get

$$f(l) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \cos n\pi$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$f(l) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{--- } (3)$$

$$\boxed{\because (-1)^n(-1)^n = 1}$$

To find $f(l)$:

$f(x) = x^2$ in $-l < x < l$

$x = l$ is end point of $(-l, l)$

$$f(l) = l^2$$

$$f(-l) = (-l)^2 = l^2$$

$$f(l) = f(-l) = l^2$$

Here $x = l$ is a point of continuity

$$\therefore f(l) = l^2$$

Substitute $f(x) = l^2$ in (3), we get

$$l^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = l^2 - \frac{l^2}{3}$$

$$= \frac{2l^2}{3}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2l^2}{3}\frac{\pi^2}{4l^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots = \frac{\pi^2}{6}$$

(ii) To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$.

Put $x=0$ in (2) we get

$$f(0) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}. \quad \text{—— (4)}$$

To find $f(0)$:

$$f(x) = x^2$$

$\therefore x=0$ lies inside $(-l, l)$

Here $x=0$ is a point of continuity

$\therefore f(0) = 0$

Substitute $f(0) = 0$ in (4) we get

$$0 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-l^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-l^2}{3}\frac{\pi^2}{4l^2}$$

$$(-1)\frac{1}{1^2} + \frac{1}{2^2} + (-1)\frac{1}{3^2} + - \cdots = \frac{-\pi^2}{12}$$

Ans: $\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}}$

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② Find the Fourier series $f(x) = |x|, -\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \frac{\pi^2}{8}$

Solution:

$f(x)$ is defined in $(-\pi, \pi)$

In $(-\pi, \pi)$ check whether $f(x)$ is even or odd

$$f(x) = |x|$$

$$f(-x) = |-x| = |x| = f(x)$$

$$f(-x) = f(x)$$

$\therefore f(x)$ is an even function

$\boxed{\therefore b_n = 0}$

The Fourier series for $f(x)$ in $(-\pi, \pi)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx \qquad \boxed{\because |x| = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}}$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{\pi^2}{2}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = \begin{cases} \frac{-4}{\pi n^2} & ; n = 1, 3, 5, \dots \\ 0 & ; n = 2, 4, 6, \dots \end{cases}}$$

Substituting the values of a_0, a_n in (1), we get.

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3}^{\infty} \frac{-4}{\pi n^2} \cos nx \quad \text{— (2)}$$

i) To deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Put $x=0$ in (2), we get

$$f(0) = \frac{\pi}{2} + \sum_{n=1,3}^{\infty} \frac{-4}{\pi n^2}$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{— (3)}$$

To find $f(0)$:

$x=0$ lies inside $(-\pi, \pi)$.

Here $x=0$ is a point of continuity

$f(x) = |x|$

$f(0) = |0| = 0$

Substitute $f(0) = 0$ in (3), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$-\frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n^2} = \frac{-\pi}{2}.$$

$$\sum_{n=1,3}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}}$$

Hence proved