Properties of lattices:

Let $(L, \wedge, \vee)$ be a given lattice. Then for any $a, b, c \in L$.

1). Idempotent law

$a \wedge a = a$ and $a \vee a = a$

2). Commutative law

$a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

3). Associative law

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$

4). Absorption law

$a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Proof:

1). Idempotent law:

Now $a \vee a = LUB(a, a) = LUB(a) = a$

$a \vee a = a$

and $a \wedge a = GLB(a, a) = GLB(a) = a$

$a \wedge a = a$

2). Commutative law:

Now $a \vee b = LUB(a, b) = LUB(b, a) = b \vee a$

and $a \wedge b = GLB(a, b) = GLB(b, a) = b \wedge a$

3). Associative:

Let $a \vee (b \vee c) = d \rightarrow (1)$

$(a \vee b) \vee c = e \rightarrow (2)$

$(1) \Rightarrow d$ is LUB of $(a, b \vee c)$

$\Rightarrow d \geq a$ and $d \geq b \vee c \longrightarrow (3)$

WKT $b \vee c$ is LUB of (b, c)

$b \vee c \geq b$ and $b \vee c \geq c \longrightarrow (4)$

$$\left.\begin{array}{l} d \geq a \\ d \geq b \\ d \geq c \end{array}\right\} \longrightarrow (5)$$

(5) $\Rightarrow$ d is an UB of (a, b) and $d \succeq c$
$d \succeq LUB(a, b)$
$d \succeq a \vee b$. and $d \succeq c$
$\therefore$ d is an UB of $(a \vee b, c)$
Since e is LUB of $(a \vee b, c)$
$d \succeq e$
Illy $e \succeq d$
$\therefore$ $e = d.$ $\Rightarrow$ $a \vee (b \vee c) = (a \vee b) \vee c$

4). Absorption law:
$a \vee (a \wedge b) = a$
Since $a \wedge b$ is GLB of $\{a, b\}$
$a \wedge b \leq a \rightarrow (1)$
obviously $a \leq a \rightarrow (2)$
(1) and (2), $a \vee (a \wedge b) \leq a \rightarrow (3)$
By defn. of LUB, we've
$a \leq a \vee (a \wedge b) \rightarrow (4)$

$a \vee (a \wedge b) = a.$

Theorem: 1 Isotonicity of law or property
Let $(L, \wedge, \vee)$ be a gvn. lattice for any $a, b, c \in L$. then prove that

$b \leq c \Rightarrow \begin{cases} 1). & a \wedge b \leq a \wedge c \\ 2). & a \vee b \leq a \vee c \end{cases}$

Proof:
Given $b \leq c$
GLB $\{b, c\} = b \wedge c = b$ (since $b \leq c \Leftrightarrow b \vee c = c$
LUB $(b, c) = b \vee c = c$ $\Leftrightarrow b \wedge c = b$)

To prove:
1). $a \wedge b \leq a \wedge c$.
It is enough to prove that GLB$(a \wedge b, a \wedge c) = a \wedge b$
ie., $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

Now

$$(a \wedge b) \wedge (a \wedge c) = a \wedge (b \wedge a) \wedge c$$
$$= a \wedge (a \wedge b) \wedge c$$
$$= (a \wedge a) \wedge (b \wedge c)$$
$$= a \wedge (b \wedge c)$$
$$= a \wedge b$$

$\therefore a \wedge b \leq a \wedge c$

2). $a \vee b \leq a \vee c$

It is enough to prove $LUB(a \vee b, a \vee c) = a \vee c$

$$(a \vee b) \vee (a \vee c) = a \vee c$$

Now

$$(a \vee b) \vee (a \vee c) = a \vee (b \vee a) \vee c$$
$$= a \vee (a \vee b) \vee c$$
$$= (a \vee a) \vee (b \vee c)$$
$$= a \vee (b \vee c)$$
$$= a \vee c$$

$\therefore a \vee b \leq a \vee c$

Theorem: 3 Distributive inequality

Let $(L, \wedge, \vee)$ be a given lattice for any $a, b, c \in L$, the following inequality holds.

i). $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

ii). $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Proof:

i). $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

From the defn. of LUB, it is obvious that

$$a \leq a \vee b \longrightarrow (1)$$

and $b \wedge c \leq b \leq a \vee b$

$$\Rightarrow b \wedge c \leq a \vee b \longrightarrow (2)$$

From (1) and (2),

$a \vee b$ is an upper bound of $\{a, b \wedge c\}$

Hence $a \vee b \geq a \vee (b \wedge c) \rightarrow (A)$

From the defn. of LUB, it is obvious that

$a \leq a \vee c \rightarrow (3)$

and $b \wedge c \leq c \leq a \vee c$

$\Rightarrow b \wedge c \leq a \vee c \rightarrow (4)$

From (3) and (4), $a \vee c$ is an UB of $\{a, b \wedge c\}$

Hence $a \vee c \geq a \vee (b \wedge c) \rightarrow (B)$

From (A) and (B), we've $a \vee (b \wedge c)$ is a LB of $\{a \vee b, a \vee c\}$.

$\therefore a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Hence proved (i).

ii). $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

WKT $a \geq a \wedge b \longrightarrow (1)$

and $b \vee c \geq b \geq a \wedge b$

$b \vee c \geq a \wedge b \longrightarrow (2)$

From (1) and (2), $a \wedge b$ is an LB of $\{a, b \vee c\}$

Hence $a \wedge b \leq a \wedge (b \vee c) \rightarrow (C)$

WKT

$a \geq a \wedge c \longrightarrow (3)$

and $b \vee c \geq c \geq a \wedge c$

$b \vee c \geq a \wedge c \longrightarrow (4)$

From (3) & (4), $a \wedge c$ is an LB of $\{a, b \vee c\}$

Hence $a \wedge c \leq a \wedge (b \vee c) \longrightarrow (D)$

From (c) and (d), we've $a \wedge (b \vee c)$ is an UB of $\{a \wedge b, a \wedge c\}$.

$\therefore a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Hence proved (ii).