Basic Field equations - Rectangular waveguides

$$\nabla \times H = j\omega \varepsilon E$$

$$\nabla \times E = -j\omega \mu H$$

$$\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon \left[E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z} \right]$$

$$\cancel{\vec{a_x}} \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] = j\omega\varepsilon E_x \cancel{\vec{a_x}} \rightarrow (1)$$

$$-\cancel{\vec{a_y}} \cdot \left[\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right] = j\omega\varepsilon E_y \cancel{\vec{a_y}} \rightarrow (2)$$

$$\cancel{\vec{a_z}} \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] = j\omega\varepsilon E_z \cancel{\vec{a_z}} \rightarrow (3)$$

Subs $\frac{\partial}{\partial z} = -\bar{\gamma}$ in eqns (1), (2) & (3)

$$\frac{\partial H_z}{\partial y} + \bar{\gamma} H_y = j\omega\varepsilon E_x \rightarrow (4)$$

$$+\frac{\partial H_z}{\partial x} + \bar{\gamma} H_x = -j\omega\varepsilon E_y \rightarrow (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \rightarrow (6)$$

IIIly from $\nabla \times E = -j\omega\mu H$

$$\frac{\partial E_z}{\partial y} + \bar{\gamma} E_y = -j\omega\mu H_x \rightarrow (7)$$

$$\frac{\partial E_z}{\partial x} + \bar{\gamma} E_x = +j\omega\mu H_y \rightarrow (8)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \rightarrow (9)$$

from eq (4)

$$H_y = \frac{1}{\bar{\gamma}}\left[j\omega\varepsilon E_x - \frac{\partial H_z}{\partial y}\right]$$

Subs in eq (8)

$$\frac{\partial E_z}{\partial x} + \bar{\gamma}E_x = +j\omega\mu\left[\frac{1}{\bar{\gamma}}\left(j\omega\varepsilon E_x - \frac{\partial H_z}{\partial y}\right)\right]$$

Simplifying

$$\frac{\partial E_z}{\partial x} + \bar{\gamma}E_x = +\frac{j\omega\mu}{\bar{\gamma}} \times j\omega\varepsilon E_x - \frac{j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y}$$

$$\frac{\partial E_z}{\partial x} + \bar{\gamma}E_x = \frac{-\omega^2\mu\varepsilon E_x}{\bar{\gamma}} - \frac{j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y}$$

$$\bar{\gamma}E_x + \frac{\omega^2\mu\varepsilon E_x}{\bar{\gamma}} = \frac{-j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}$$

$$\frac{\bar{\gamma}^2E_x + \omega^2\mu\varepsilon E_x}{\bar{\gamma}} = \frac{-j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}$$

$$E_x\left[\frac{\gamma^2 + \omega^2\mu\varepsilon}{\bar{\gamma}}\right] = -\frac{j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}$$

$$h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon$$

$$\therefore \quad E_x\frac{h^2}{\bar{\gamma}} = \frac{-j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}$$

$$E_x = \frac{\bar{\gamma}}{h^2}\left[\frac{-j\omega\mu}{\bar{\gamma}}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}\right]$$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x} \rightarrow (10)$$

$$H_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

For TE waves ($E_z = 0$)

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_x = -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$H_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x}$$

$$H_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y}$$

TM waves ($H_z = 0$)

$$E_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

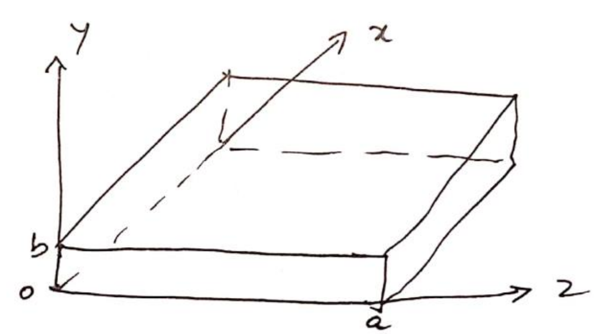
$$H_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

E_z & $H_z = 0$, No fields within the guide.

Boundary conditions

$E_x = E_z = 0$ at $y = 0$ & $y = b$

$E_y = E_z = 0$ at $x = 0$ & $x = a$



(Fig) A Rectangular waveguide

$a \rightarrow$ width of the guide
$b \rightarrow$ height of the guide.

TM waves

Product soln. method

$$E_z(x,y,z) = E_z^\circ(x,y)e^{-\bar{\gamma} z} \rightarrow (1)$$

$E_z^\circ = X\,Y$

$X \rightarrow$ function of x alone

$Y \rightarrow$ function of y alone

Wave eqn.

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2\mu\epsilon E_z \rightarrow (2)$$

$$\frac{\partial^2}{\partial z^2} = \bar{\gamma}^2$$

subs in eq (2) & interms of E_z°

$$\frac{\partial^2 E_z^\circ}{\partial x^2} + \frac{\partial^2 E_z^\circ}{\partial y^2} + \bar{\gamma}^2 E_z^\circ = -\omega^2\mu\epsilon E_z^\circ \rightarrow (3)$$

subs $E_z^\circ = XY$ in eq (3)

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (\bar{\gamma}^2 + \omega^2\mu\epsilon)XY = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + h^2 XY = 0$$

$\div$ by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + h^2 = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \rightarrow (4)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = A^2 \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + (h^2 - A^2) = 0$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -A^2 \rightarrow (5) \qquad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + B^2 = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -B^2 \rightarrow (6)$$

Soln. of eq (6) & (5)

$$X = C_1 \cos Bx + C_2 \sin Bx \rightarrow (7)$$

$$Y = C_3 \cos Ay + C_4 \sin Ay \rightarrow (8)$$

$$\therefore E_z^{o} = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay) \rightarrow (9)$$

Subs B.C 1

$E_z^{o} = 0$ at $x = 0$ in eq (9)

$$\cancel{E_z^{o} = (C_1 \cos 0)(C_3 \cos 0)}$$

$$E_z^{o} = (C_1 \cos 0 + 0)(C_3 \cos Ay + C_4 \sin Ay)$$

$$E_z^{o} = C_1 (C_3 \cos Ay + C_4 \sin Ay)$$

$$\therefore \boxed{C_1 = 0}$$

Subs in eq (9)

$$E_z^{o} = (C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay) \rightarrow (10)$$

Subs B.C 2 $E_z^{o} = 0$ at $y = 0$ in eq (10)

$$E_z^{o} = (C_2 \sin Bx)(C_3)$$

To satisfy B.C either C_2 (or) $C_3 = 0$

But $C_2 = 0$ in eq (10) makes all the fields zero

So $\boxed{C_3 = 0}$

$\therefore$ Eq (10) $\Rightarrow$ $E_z^{o} = C_2 \sin Bx \, C_4 \sin Ay$

$$E_z^o = C_2 C_4 \sin Bx \sin Ay \rightarrow (11)$$

B.C 3 $\quad E_z^o = 0$ at $x = a$ in eq (11)

$$E_z^o = C_2 C_4 \sin Ba \sin Ay$$

To satisfy the B.C, $B = \frac{m\pi}{a}$, $m = 1, 2, 3 \ldots$

$$\therefore \; E_z^o = C_2 C_4 \sin\left(\frac{m\pi}{a}\right) x \sin Ay$$

B.C 4 $\quad E_z^o = 0$ at $y = b$ in eq (11)

$$E_z^o = C_2 C_4 \sin Bx \sin Ab$$

To satisfy the B.C 4, $A = \frac{n\pi}{b}$, $n = 1, 2, \ldots$

Subs in eq (11)

$$E_z^o = C_2 C_4 \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right)$$

$$E_z = C_2 C_4 \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-\bar{\gamma} z}$$

subs $\bar{\gamma} = j\bar{\beta}$ & subs E_z in the basic field eqns. other fields are obtained,

$$E_x^o = \frac{-j\bar{\beta} C}{h^2} B \cos Bx \sin Ay$$

$$E_y^o = \frac{-j\bar{\beta} C}{h^2} A \sin Bx \cos Ay$$

$$H_x^o = \frac{j\omega\varepsilon C}{h^2} A \sin Bx \cos Ay$$

$$H_y^o = -\frac{j\omega\varepsilon}{h^2} B \cos Bx \sin Ay$$

where $B = \frac{m\pi}{a}$, $A = \frac{n\pi}{b}$, $C = C_2 C_4$

TE waves

By product soln. method

$$H_z^{\circ} = (C_1 \cos Bx \; C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay) \rightarrow (1)$$

B.C.

$E_y = 0$ at $x = 0$ & $x = a$

$E_x = 0$ at $y = 0$ & $y = b$

~~Ex~~ Diff eq (1) w.r.t y

$$\frac{\partial H_z^{\circ}}{\partial y} = (C_1 \cos Bx + C_2 \sin Bx)[-C_3 A \sin Ay + C_4 A \cos Ay]$$

Subs B.C (i) $E_x = 0$ at $y = 0$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$= \frac{-j\omega\mu}{h^2}\left[(C_1 \cos Bx + C_2 \sin Bx)(-C_3 A \underbrace{\sin Ay}_{0} + C_4 A \cos Ay\right]$$

$\therefore \boxed{C_4 = 0}$

$$\cancel{E_x = \frac{-j\omega\mu}{h^2}(C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay)}$$

→ subs in eq (1)

$$H_z = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay)$$

Diff w.r.t. x

$$\frac{\partial H_z}{\partial x} = (-C_1 B \sin Bx + C_2 B \cos Bx)(C_3 \cos Ay)$$

$E_y = 0$ at $x = 0$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$= \frac{j\omega\mu}{h^2}\left[-C_1 B \underbrace{\sin Bx}_{0} + C_2 B \cos Bx\right][C_3 \cos Ay]$$

$$E_y = \frac{j\omega\mu}{h^2} (C_2 B)(C_3 \cos Ay)$$

$$\therefore \boxed{C_2 = 0}$$

$$\therefore \boxed{H_z^0 = C_1 \cos Bx \, C_3 \cos Ay}$$

$$B = \frac{m\pi}{a}, \quad A = \frac{n\pi}{b}$$

$$C_1 C_3 = C$$

$$\boxed{H_z^0 = C \cos Bx \cos Ay}$$

Other fields are

$$H_x^0 = \frac{j\bar{\beta}}{h^2} C B \sin Bx \cos Ay$$

$$H_y^0 = \frac{j\bar{\beta}}{h^2} C A \cos Bx \sin Ay$$

$$E_x^0 = \frac{j\omega\mu}{h^2} C A \cos Bx \sin Ay$$

$$E_y^0 = -\frac{j\omega\mu}{h^2} C B \sin Bx \cos Ay$$