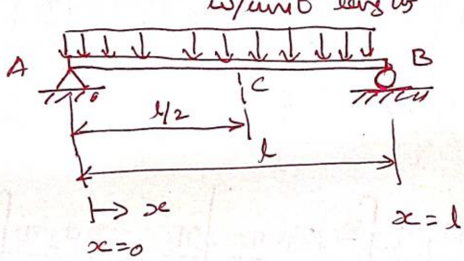
using trigonometric series – two term trial function

i. Derive the expression for deflection and bending moment in a simply supported beam of span length l, subjected to uniformly distributed load over entire span using two term trigonometric trial function. Also find the deflection and moment at mid span and compare with exact solution. Use Rayleigh-Ritz method. [AU 2008]



Two term trigonometric train function such as

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} \quad \text{---- (1)}$$

The boundary condition for the uniformly distributed loaded beam are

$y = 0$ at $x = 0$ and at $x = l$

Since the above selected trial function satisfies the boundary conditions, this function is correct.

No applying the total potential energy concept we get,

$$\pi = U - W$$

where U = Internal strain energy

W = workdone by the external force

Strain energy for a beam is given by

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \quad \text{—③}$$

Now differentiating the deflection function [i.e. Equation ①] two times we get

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

$$\frac{dy}{dx} = a_1\left[\cos\frac{\pi x}{l}\right]\frac{\pi}{l} + a_2\left[\cos\frac{3\pi x}{l}\right]\frac{3\pi}{l}$$

$$= \frac{a_1\pi}{l}\cos\frac{\pi x}{l} + \frac{3a_2\pi}{l}\cos\frac{3\pi x}{l}$$

$$\frac{d^2y}{dx^2} = \frac{a_1\pi}{l}\left[-\sin\frac{\pi x}{l}\right]\frac{\pi}{l} + \frac{3a_2\pi}{l}\left[-\sin\frac{3\pi x}{l}\right]\frac{3\pi}{l}$$

$$= -\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

Hence

$$\left[\frac{d^2y}{dx^2}\right]^2 = \left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2$$

$$= \left[\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]^2$$

$$= \frac{\pi^4}{l^4}\left[a_1\sin\frac{\pi x}{l} + 9a_2\sin\frac{3\pi x}{l}\right]^2$$

②

$$= \frac{\pi^4}{l^4}\left[a_1^2 \sin^2\frac{\pi x}{l} + 81 a_2^2 \sin^2\frac{3\pi x}{l} + 18 a_1 a_2 \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}\right]$$

using the relation $(a+b)^2 = a^2+b^2+2ab$

Now, the strain energy equation (i.e. Eqn ③) implies.

$$U = \frac{EI}{2}\int_0^l \left[\frac{d^2y}{dx^2}\right]^2 dx$$

$$= \frac{EI}{2}\int_0^l \frac{\pi^4}{l^4}\left(a_1^2 \sin^2\frac{\pi x}{l} + 81 a_2^2 \sin^2\frac{3\pi x}{l} + 18 a_1 a_2 \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}\right) dx$$

$$= \frac{EI\pi^4}{2l^4}\int_0^l \left[a_1^2 \sin^2\frac{\pi x}{l} + 81 a_2^2 \sin^2\frac{3\pi x}{l} + 18 a_1 a_2 \sin\frac{\pi x}{l}\cdot\sin\frac{3\pi x}{l}\right] \quad \text{--- (4)}$$

Since the simultaneous integration of all the three inner terms of equation (4) is slighty difficult, let us integrate them individually

Now

$$\int_0^l a_1^2 \sin^2\frac{\pi x}{l}\, dx = a_1^2 \int_0^l \frac{1}{2}\left[1 - \cos\frac{2\pi x}{l}\right] dx$$

using the relation $\sin^2 A = \frac{1}{2}(1-\cos 2A)$

$$= \frac{a_1^2}{2}\left[x - \frac{\sin\frac{2\pi x}{l}}{2\pi/l}\right]_0^l$$

$$= \frac{a_1^2}{2}\left[(l-0) - \frac{l}{2\pi}(\sin 2\pi - \sin 0)\right]$$

$\sin 2\pi = 0$

$\sin 0 = 0$

$$= \frac{a_1^2 l}{2} \quad \text{--- (5)}$$

Similarly

$$\int_0^l 81 a_2^2 \sin^2 \frac{3\pi x}{l}\, dx$$

$$= 81 a_2^2 \int_0^l \frac{1}{2}\left[1 - \cos\frac{6\pi x}{l}\right] dx$$

$$= \frac{81 a_2^2}{2}\left[x - \frac{\sin\frac{6\pi x}{l}}{6\pi/l}\right]_0^l$$

$$= \frac{81 a_2^2}{2}\left[(l-0) - \frac{l}{6\pi}(\sin 6\pi - \sin 0)\right]$$

$\sin 6\pi = 0$, $\sin 0 = 0$

$$= \frac{81 a_2^2 l}{2} \quad \text{—} (6)$$

and

$$\int_0^l 18 a_1 a_2 \sin\frac{\pi x}{l} \sin\frac{3\pi x}{l}\, dx$$

$$= 18 a_1 a_2 \int_0^l \sin\frac{3\pi x}{l} \cdot \sin\frac{\pi x}{l}\, dx$$

$$= 18 a_1 a_2 \int_0^l \frac{1}{2}\left(\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l}\right) dx$$

(using the relation $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$)

$$= 9a_1a_2\left[\frac{\sin\frac{2\pi x}{l}}{\left(\frac{2\pi}{l}\right)} - \frac{\sin\frac{4\pi x}{l}}{\left(\frac{4\pi}{l}\right)}\right]_0^l$$

$$= 9a_1a_2\left[\frac{l}{2\pi}(\sin 2\pi - \sin 0) - \frac{l}{4\pi}(\sin 4\pi - \sin 0)\right] = 0 \Rightarrow (E)$$

$\because$ $\sin 2\pi = 0$, $\sin 4\pi = 0$, $\sin 0 = 0$

Substituting the values of equation (5), (6) and (7) in equation (4), $U = \frac{EI}{2}\int\left(\frac{d^2y}{dx^2}\right)^2 dx$ ... we get

$$U = \frac{EI\pi^4}{2l^4}\left[\frac{a_1^2 l}{2} + \frac{81 a_2^2 l}{2}\right]$$

$$= \frac{EI\pi^4}{4l^3}\left[a_1^2 + 81 a_2^2\right]$$

Now, the work done by the external force is given by

$$W = \int_0^l w\,y\,dx$$

$$= \int_0^l w\left[a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}\right] dx$$

$$= w\left[\int_0^l a_1 \sin\frac{\pi x}{l}\,dx + \int_0^l a_2 \sin\frac{3\pi x}{l}\,dx\right]$$

$$= w\left\{\left[a_1 \frac{-\cos\frac{\pi x}{l}}{\pi/l}\right]_0^l + \left[a_2 \frac{-\cos\frac{3\pi x}{l}}{3\pi/l}\right]_0^l\right\}$$

⑤

$$= W\left[-\frac{a_1 l}{\pi}(\cos\pi - \cos 0) - \frac{a_2 l}{3\pi}(\cos 3\pi - \cos 0)\right]$$

$$= W\left[-\frac{a_1 l}{\pi}(-1-1) - \frac{a_2 l}{3\pi}(-1-1)\right] \quad \left[\because \cos\pi = -1,\ \cos 3\pi = -1,\ \cos 0 = 1\right]$$

$$= W\left[\frac{2a_1 l}{\pi} + \frac{2a_2 l}{3\pi}\right]$$

$$= \frac{2Wl}{\pi}\left(a_1 + a_2/3\right) \quad -(9)$$

Substituting the values of equation (8) & (9) in equation (2) we get,

Total potential energy, $\pi = U - W$

$$= \frac{EI\pi^4}{4l^3}\left(a_1^2 + 81a_2^2\right) - \frac{2Wl}{\pi}\left(a_1 + \frac{a_2}{3}\right)$$

For minimum potential energy, the following condition must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{\partial\pi}{\partial a_1} = \frac{EI\pi^4}{4l^3}(2a_1) - \frac{2Wl}{\pi} = 0 \Rightarrow a_1 = \frac{4Wl^4}{EI\pi^5}$$

$$\frac{\partial\pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2Wl}{3\pi} = 0 \Rightarrow a_2 = \frac{4Wl^4}{243EI\pi^5}$$

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Year & Branch	:	**III Aerospace**	Semester	:	**V**
Course	:	**19ASB302 – Finite Element Method for Aerospace**			

Unit: 1

Hence the deflection

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

$$= \frac{4wl^4}{EI\pi^5}\sin\frac{\pi x}{l} + \frac{4wl^4}{243 EI\pi^5}\sin\frac{3\pi x}{l}$$

Maximum deflection at $x = l/2$ is obtained as

$$y_{max} = \frac{4wl^4}{EI\pi^5}\sin\left(\frac{\pi}{l}\times\frac{l}{2}\right) + \frac{4wl^4}{243EI\pi^5}\sin\left(\frac{3\pi}{l}\times\frac{l}{2}\right)$$

$$= \frac{4wl^4}{EI\pi^5}\sin\frac{\pi}{2} + \frac{4wl^4}{243EI\pi^5}\sin\left(\frac{3\pi}{2}\right)$$

$$= \frac{4wl^4}{EI\pi^5} - \frac{4wl^4}{243EI\pi^5} \qquad \left[\because \sin\frac{\pi}{2} = 1,\ \sin 3\pi/2 = -1\right.$$

$$= \frac{4wl^4}{EI\pi^5}\left[1 - \frac{1}{243}\right]$$

$$= \frac{3.984\,wl^4}{EI\pi^5}$$

$$= \frac{wl^4}{EI\left[\frac{\pi^5}{3.984}\right]} = \frac{wl^4}{76.81\,EI}$$

$$\boxed{y_{max} = \frac{wl^4}{76.81\,EI}}$$

This is almost equal to the exact value of maximum deflection which is

$$y_{max} = \frac{5}{384}\frac{wl^4}{EI} = \frac{wl^4}{76.8EI}$$

Prepared: Dr. M. Subramanian/Professor & Head Aerospace Engineering

Determination of bending moment at mid-span.

For a beam, the bending moment is given by the expression as

Bending moment, $M = EI \frac{d^2y}{dx^2}$

From the previous derivation we know that

$$\frac{d^2y}{dx^2} = -\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

Hence

$$M = EI\left(-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right)$$

$$= -\frac{EI\pi^2}{l^2}\left(a_1\sin\frac{\pi x}{x} + 9a_2\sin\frac{3\pi x}{l}\right)$$

Substituting the values of a_1 and a_2 in the above equation, we get

$$M = -\frac{EI\pi^2}{l^2}\left(\frac{4wl^4}{EI\pi^5}\sin\frac{\pi x}{l} + 9\times\frac{4wl^4}{243EI\pi^5}\sin\frac{3\pi x}{l}\right]$$

$$= -\left(\frac{EI\pi^2}{l^2}\right)\left(\frac{4wl^4}{EI\pi^5}\right)\left[\sin\frac{\pi x}{l} + \frac{9}{243}\sin\frac{3\pi x}{l}\right]$$

Bending moment at $x = \frac{l}{2}$ is obtained as

Maximum bending moment

$$M_{max} = -\frac{4wl^2}{\pi^3}\left[\sin\left(\frac{\pi}{l}\times l/2\right) + \frac{9}{243}\sin\left(\frac{3\pi}{l}\times\frac{l}{2}\right)\right]$$

$$= \frac{-4wl^2}{\pi^3}\left[\sin\pi/2 + 9/243\sin 3\pi/2\right]$$

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$$= -\frac{4wl^2}{\pi^3}\left(1 - \frac{9}{243}\right) \quad \left[\because \sin\pi/2 = 1 \;\&\; \sin 3\pi/2 = -1\right]$$

$$= -\frac{3.852wl^2}{\pi^3} = \frac{-wl^2}{\left(\pi^3/3.852\right)}$$

$$= -\frac{wl^2}{8.05}$$

Negative sign is due to downward loading

$$M_{max} = \frac{wl^2}{8.05}$$

This is almost equal to the exact value of maximum bending moment which $M_{max} = wl^2/8$