Faculty Name	:	**Dr.M.Subramanian, Prof & Head/ Aerospace**	Academic Year	:	**2024-2025 (Odd)**
Year & Branch	:	**III Aerospace**	Semester	:	**V**
Course	:	**19ASB302 – Finite Element Method for Aerospace**			

Unit: 1

A cantilever beam of length L is loaded with a point at free end. Find the maximum deflection and bending moment using Rayleigh-Ritz method using the function

$$y = A\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right].$$

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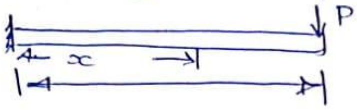

(i) To find maximum deflection:

The deflection for the above beam is

$$y = A\left\{1 - \cos\frac{\pi x}{2L}\right\} \quad —(1)$$

The boundary conditions for this beam are at $x=0$, the deflection $y=0$ and the slope $dy/dx = 0$

From eqn ①, $y_{(at\ x=0)} = A\{1-\cos 0\} = 0$

$\cos 0 = 1$

Also Eqn (1) implies $\frac{dy}{dx} = A\left\{0 + \frac{\pi}{2L}\sin\frac{\pi x}{2L}\right\}$

At $x=0$, $dy/dx = A\left\{0 + \frac{\pi}{2L}\sin 0\right\} = 0$

Since the boundary condition are satisfied by the trial deflection function, it is the correct trial function.

T.P.E $\rightarrow \Pi = U - W \quad —②$

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$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \quad \text{—③}$$

To find the value of strain energy, let us differentiate the deflection function two times as follows.

$$\frac{dy}{dx} = A\left\{0 - \left[-\sin\frac{\pi x}{2L}\right]\frac{\pi}{2L}\right\} = \frac{A\pi}{2L}\sin\frac{\pi x}{2L}$$

$$\frac{d^2y}{dx^2} = \frac{A\pi}{2L}\left(\cos\frac{\pi x}{2L}\right)\frac{\pi}{2L} = \frac{A\pi^2}{4L^2}\cos\frac{\pi x}{2L}$$

Now

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[\frac{A\pi^2}{4L^2}\cos\frac{\pi x}{2L}\right]^2 = \frac{A^2\pi^4}{16L^4}\cos^2\frac{\pi x}{2L}$$

$$= \frac{A^2\pi^4}{16L^4}\left\{\frac{1+\cos\frac{\pi x}{L}}{2}\right\} \quad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right] \text{ using the relation}$$

The strain energy

$$U = \frac{EI}{2}\int_0^L \frac{A^2\pi^4}{16L^4}\left[\frac{1}{2}\left(1+\cos\frac{\pi x}{L}\right)\right]dx$$

$$= \frac{EI\,A^2\pi^4}{64L^4}\int_0^L \left[1+\cos\frac{\pi x}{L}\right]dx$$

$$= \frac{EI\,A^2\pi^4}{64L^4}\left[x + \frac{L}{\pi}\sin\frac{\pi x}{L}\right]_0^L$$

$$= \frac{EI\,A^2\pi^4}{64L^4}\left[(L-0) + \frac{L}{\pi}(\sin\pi - \sin 0)\right] \quad \because \sin\pi = 0,\ \sin 0 = 0$$

$$= \frac{EI\,A^2\pi^4}{64L^4}\left[(L-0) + \frac{L}{\pi}(0-0)\right] = \frac{EI\,A^2\pi^4}{64L^3} \quad \text{—④}$$

work done $W = P \cdot y_{max} = P\, y$ at $x = l$

$$= P\left[A\left(1-\cos\frac{\pi x}{2L}\right)\right]_{at\ x=l}$$

$$= P\left[A\left(1-\cos\frac{\pi}{2}\right)\right] = PA \quad \text{--- (5)} \qquad \because \cos\frac{\pi}{2}=0$$

Substituting the values of equation (4) & (5) in equation (2), we get,

$$\Pi = U - W = \frac{EI A^2 \pi^4}{64 L^3} - PA$$

For minimum potential energy condition, $\frac{\partial \Pi}{\partial A} = 0$

$$\frac{\partial \Pi}{\partial A} = \frac{2 EI A \pi^4}{64 L^3} - P = 0$$

which implies $A = \frac{32 PL^3}{EI\pi^4}$ which is the value of Ritz coefficient A.

Hence the maximum deflection at the free end of the cantilever beam is given by

$$y_{max} = A\left(1-\cos\frac{\pi x}{2L}\right)_{at\ x=L}$$

$$= \frac{32 PL^3}{EI\pi^4}\left(1-\cos\frac{\pi}{2}\right) \qquad \because \cos\frac{\pi}{2}=0$$

$$= \frac{32 PL^3}{EI\pi^4}$$

$$y_{max} = \frac{PL^3}{\left(\frac{\pi^4}{32}\right)EI} = \frac{PL^3}{3.04\, EI} \quad \text{——— (6)}$$

(3)

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DEPARTMENT OF AEROSPACE ENGINEERING

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the max def referred in eqn (2) is the approximate but almost equal to actual value of max def which is

$$y_{max} = \frac{PL^3}{3E}$$

(ii) To find the maximum bending moment.

Bending moment is given by the expression

$$M = E\frac{d^2y}{dx^2} = EI\frac{A\pi^2}{4L^2}\cos\frac{\pi x}{2L}$$

Bending moment is maximum at the fixed end i.e at $x=0$

Hence

$$M_{max} = \left[EI\frac{A\pi^2}{4L^2}\cos\frac{\pi x}{2L}\right]_{at\ x=0}$$

$$= EI\frac{A\pi^2}{4L^2}\cos 0 = EI\frac{A\pi^2}{4L^2} \qquad \because \cos 0 = 1$$

$$= \frac{EI\pi^2}{4L^2}\times\frac{32PL^3}{EI\pi^4} \qquad \left[\text{Substituting the value of } A = \frac{32PL^3}{EI\pi^4}\right]$$

$$= \frac{8PL}{\pi^2} = \frac{PL}{\left[\frac{\pi^2}{8}\right]} = \frac{PL}{1.23}$$

The above value is the approximate value of maximum bending moment and it is close to the actual maximum bending moment which is $M_{max} = PL$