Path:

A path in a graph is a sequence $v_1, v_2, \ldots, v_k$ of vertices, each adjacent to the next.

Length of the path:

The No. of edges appearing in the sequence of path is called the length of the path.

Circuits:

cycle / circuit | A path which originates & ends in the same node is called a cycle.

A closed path in which all the edges are distinct is called a circuit.

Cycle:

A circuit in which all the vertices are distinct is a cycle.

Connected graph:

An directed graph is said to be connected if any pair of nodes are reachable from one another. i.e., there is a path b/w any pair of nodes.

Strongly connected:

A simple digraph is said to be strongly connected if for any pair of nodes of the graph are reachable from both the nodes of the pair are reachable from one another.

Weakly connected:

A simple digraph is said to be weakly connected if it is connected as an undirected graph in which the direction of the edges is neglected.

Unilaterally connected:

A simple digraph is said to be unilaterally connected, if for any pair of nodes of the graph atleast one of the nodes of the pair is reachable from the other node.

Note: 1). A UC is WC but a WC is not necessarily UC.

2). A SC is both U & WC.

Theorem 4:

A simple graph with n vertices and k components cannot have more than $\frac{(n-k)(n-k+1)}{2}$ edges.

(or)

A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:

Let $n_1, n_2, \dots, n_k$ be the no. of vertices in each of k components of the graph G.

Then $n_1 + n_2 + \dots + n_k = n = |V(G)|$

$$\sum_{i=1}^{k} n_i = n \rightarrow (1)$$

Now,
$$\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1)$$
$$= \sum_{i=1}^{k} n_i - k$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

Squaring on bothsides,

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$(n_1-1)^2 + (n_2-1)^2 + \dots + (n_k-1)^2 \le n^2 + k^2 - 2nk$$

$$n_1^2 + 1 - 2n_1 + n_2^2 + 1 - 2n_2 + \dots + n_k^2 + 1 - 2n_k \le n^2 + k^2 - 2nk$$

$$(n_1^2 + n_2^2 + \dots + n_k^2) - 2n_1 - 2n_2 - \dots - 2n_k + 1 + 1 + 1 + \dots + 1 \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 - 2(n_1 + n_2 + \dots + n_k) + k \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 - 2n + k \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk + 2n - k \rightarrow (2)$$

By thm. 3, G is simple and maximum no. of edges of G in its component is $\frac{n_i(n_i - 1)}{2}$.

Maximum No. of edges of G

$$= \sum_{i=1}^{K} \frac{n_i(n_i-1)}{2}$$

$$= \sum_{i=1}^{K} \frac{n_i^2 - n_i}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{K} n_i^2 - \sum_{i=1}^{K} n_i\right]$$

$$\leq \frac{1}{2}\left[n^2 + K^2 - 2nK + 2n - K - n\right]$$

$$\leq \frac{1}{2}\left[n^2 - n + K^2 - K - 2nK + 2n\right]$$

$$\leq \frac{1}{2}\left[n^2 - 2nK + K^2 - K + n\right]$$

$$\leq \frac{1}{2}\left[(n-K)^2 + (n-K)\right]$$

maximum No. of edges in G $\leq \frac{1}{2}\left[(n-K)(n-K+1)\right]$

Theorem 5:

Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges

Proof:

Let G be a graph with n vertices and has more than $\frac{(n-1)(n-2)}{2}$ edges.

To prove G is connected.

Suppose G is not connected, then G must have atleast 2 components. Let it be G_1 and G_2.

By thm. 4, a simple graph with n vertices and K components can have atmost $\frac{1}{2}(n-K)(n-K+1)$ edges.

ie., $|E(G)| \leq \frac{1}{2}(n-2)(n-2+1)$

$|E(G)| = \frac{1}{2}(n-2)(n-1)$ which is a $\Rightarrow \Leftarrow$ to our assump.

Hence G is connected.