LINEAR NON-HOMOGENEOUS RECURRENCE EQUATIONS:

(1) Solve the recurrence equation $S(K) - 7S(K-1) + 10S(K-2) = 8K+6$, with the initial conditions $S(0)=1$, $S(1)=2$.

Given: $S(K) - 7S(K-1) + 10S(K-2) = 8K+6$ with

$S(0)=1$, $S(1)=2$.

(ie) $a_n - 7a_{n-1} + 10a_{n-2} = 8n+6$ with $a_0=1$, $a_1=2$

$\therefore$ The char. eqn is $m^2 - 7m + 10 = 0$

char. roots are $m = 2, 5$

$\therefore$ The char. eqn is $a_n = A(2)^n + B(5)^n$

Given: $RHS = 8n+6$

Finding particular solution:

put $a_n = d_0 + d_1(n)$

$a_{n-1} = d_0 + d_1(n-1)$

$a_{n-2} = d_0 + d_1(n-2)$

$\therefore$ recurrence equation becomes,

$$d_0 + d_1 n - 7(d_0 + d_1(n-1)) + 10(d_0 + d_1(n-2)) = 8n+6$$

$$d_0 + d_1 n - 7d_0 - 7d_1 n + 7d_1 + 10d_0 + 10d_1 n - 20d_1 = 8n+6$$

$$4d_0 + 4d_1 n - 13d_1 = 8n+6$$

By equating co-efficient of n & constants, we get

$4d_1 = 8$; $4d_0 - 13d_1 = 6$

$d_1 = 2$; $4d_0 - 26 = 6$

$d_0 = 8$

$\therefore$ particular soln. $a_n = 8 + 2n$

$\therefore$ General soln is a_n = char eqn + particular soln

$$a_n = A(2)^n + B(5)^n + 8 + 2n$$

Given: $a_0 = 1$

$$\Rightarrow a_0 = A(2)^0 + B(5)^0 + 8 + 2(0)$$

$$1 = A + B + 8$$

$$-7 = A + B \quad \text{——} \quad (1)$$

& also given: $a_1 = 2$

$$\Rightarrow a_1 = A(2)^1 + B(5)^1 + 8 + 2(1)$$

$$2 = 2A + 5B + 10$$

$$-8 = 2A + 5B \quad \text{——} \quad (2)$$

By solving (1) & (2) we get,

$A = -9$; $B = 2$

$$\therefore a_n = -9(2)^n + 2(5)^n + 8 + 2n$$

(2) Solve $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6$

<u>Given</u>: $a_n - 2a_{n-1} - 3a_{n-2} = 4^n + 6$

$\therefore$ The char. eqn. is $m^2 - 2m - 3 = 0$

char roots are $m = -1, 3$

$\therefore$ The char. eqn. is $a_n = A(-1)^n + B(3)^n$

<u>Given</u>: RHS $= 4^n + 6$

Finding particular solution 1:

put $a_n = d.4^n$

$a_{n-1} = d\,4^{n-1}$

$a_{n-2} = d\,4^{n-2}$

$\therefore$ recurrence equation becomes,

$d4^n - 2d4^{n-1} - 3d4^{n-2} = 4^n \cancel{+6}$

$4^n \left[d - \cancel{2}d \cdot \frac{1}{\cancel{4}_2} - 3d \cdot \frac{1}{16}\right] = 4^n$

$\frac{16d - 8d - 3d}{16} = 1 \Rightarrow d = 16/5$

$\therefore$ particular soln., $a_n = \left(\frac{16}{5}\right)4^n$

Finding particular solution 2:

put $a_n = a_{n-1} = a_{n-2} = d$

recurrence eqn. becomes, $d - 2d - 3d = 6$

$\Rightarrow d = -3/2$

$\therefore$ particular soln, $a_n = -3/2$.

$\therefore$ general soln. a_n = char. eqn + part. soln 1 + part. soln 2

$= A(3)^n + B(-1)^n + \frac{16}{5} 4^n - \frac{3}{2}$

(3) Solve the recurrence equation $a_n - 4a_{n+1} + 4a_{n-2} = 2^n + 3n$, $n \geq 2$.

Given: $a_n - 4a_{n-1} + 4a_{n-2} = 2^n + 3n$

$\therefore$ The char. eqn. is $m^2 - 4m + 4 = 0$

char. roots are $m = 2, 2$

$\therefore$ char. eqn. is $a_n = (A + Bn)(2)^n$

Given: RHS $= 2^n + 3n$

Finding particular soln. 1 : (roots are repeated)

put $a_n = dn^2(2)^n$

$a_{n-1} = d(n-1)^2(2)^{n-1}$

$a_{n-2} = d(n-2)^2(2)^{n-2}$

$\therefore$ recurrence equation becomes,

$dn^2(2)^n - 4d(n-1)^2(2)^{n-1} + 4d(n-2)^2(2)^{n-2} = 2^n$

$$(2)^n \left[dn^2 - \frac{4}{2}dn^2 + \frac{8dn}{2} - \frac{4d}{2} + \frac{4}{4}dn^2 - \frac{16dn}{4} + \frac{16d}{4}\right] = 2^n$$

$$\cancel{dn^2} - 2\cancel{dn^2} + 4\cancel{d}n - 2d + \cancel{dn^2} - 4\cancel{d}n + 4d = 1$$

$$2d = 1$$

$$d = 1/2$$

$\therefore$ particular solution, $a_n = (1/2)\, n^2 (2)^n$

Finding particular soln. 2 :

put $a_n = d_0 + d_1 n$

$a_{n-1} = d_0 + d_1(n-1)$

$a_{n-2} = d_0 + d_1(n-2)$

$\therefore$ recurrence equations becomes,

$$(d_0 + d_1 n) - 4(d_0 + d_1(n-1)) + 4(d_0 + d_1(n-2)) = 3n$$

$$d_0 + d_1 n - 4\cancel{d}_0 - 4\cancel{d}_1 n + 4d_1 + 4\cancel{d}_0 + 4\cancel{d}_1 n - 8d_1 = 3n$$

$$d_0 + d_1 n - 4d_1 = 3n$$

By equating the co-eff. of n & const. we get

$d_1 = 3$; $d_0 - 4d_1 = 0$

$d_0 = 4(3) = 12$.

$\therefore$ particular soln. 2, $a_n = 12 + 3n$

$\therefore$ general soln. is a_n = char. eqn. + particular soln.

$$= (A + Bn)(2)^n + (1/2)\, n^2 (2)^n + 12 + 3n$$