ELECTRICAL AND MECHANICAL TRANSFER FUNCTION MODELS

The control systems can be represented with a set of mathematical equations known as mathematical model. These models are useful for analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model. Design of control system means finding the mathematical model when we know the input and the output.

The following mathematical models are mostly used.

- Differential equation model
- Transfer function model
- State space model

Let us discuss the first two models in this chapter.

Differential Equation Model

Differential equation model is a time domain mathematical model of control systems. Follow these steps for differential equation model.

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

Example

Consider the following electrical system as shown in the following figure 1.3.1. This circuit consists of resistor, inductor and capacitor. All these electrical elements are connected in **series**. The input voltage applied to this circuit is v_i and the voltage across the capacitor is the output voltage v_o.

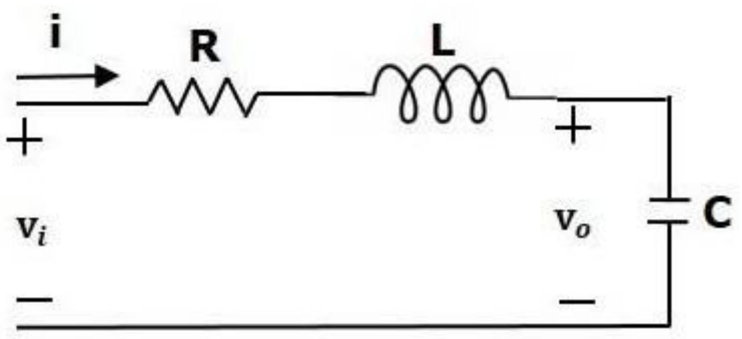

Figure 1.3.1: RLC series electrical system

[Source: "Control System Engineering" by Nagoor Kani, page : 1.20]

Mesh equation for this circuit is

$$V_i = R_i + L\frac{di}{dt} + V_o$$

Substitute, the current passing through capacitor $i = c\frac{dVo}{dt}$ in the above equation.

$$\Rightarrow V_i = RC\frac{dVo}{dt} + LC\frac{d^2V_o}{dt^2} + V_o$$

The above equation is a second order **differential equation**.

Transfer Function Model

Transfer function model is an s-domain mathematical model of control systems. The **Transfer function** of a Linear Time Invariant (LTI) system is defined as the ratio of Laplace transform of output and Laplace transform of input by assuming all the initial conditions are zero.

If x(t) and y(t) are the input and output of an LTI system, then the corresponding Laplace transforms are X(s) and Y(s).

Therefore, the transfer function of LTI system is equal to the ratio of Y(s) and X(s).

$$\text{i.e.,} transferfunction = \frac{F(s)}{K(s)}$$

The transfer function model of an LTI system is shown in the following figure1.3.2.

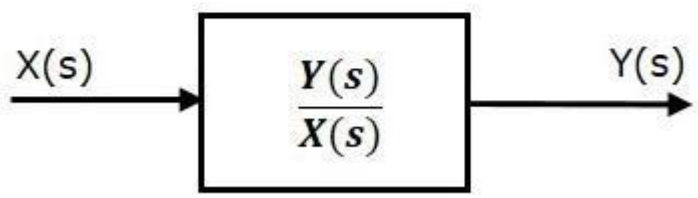

Figure 1.3.2: transfer function model

[Source: "Control System Engineering" by Nagoor Kani, page : 1.5]

Here, we represented an LTI system with a block having transfer function inside it. And this block has an input X(s) & output Y(s).

Transfer function of previous example electrical system

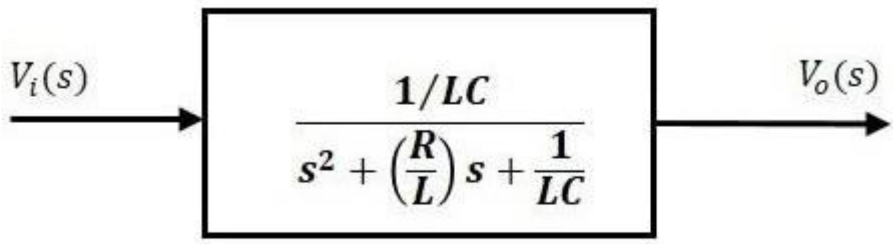

Figure 1.3.3: transfer function of RLC series electrical system

[Source: "Control System Engineering" by Nagoor Kani, page : 1.20]

Here, we show a second order electrical system with a block having the transfer function inside it. And this block has an input Vi(s) & an output Vo(s).

The differential equation modeling of mechanical systems. There are two types of mechanical systems based on the type of motion.

- Translational mechanical systems
- Rotational mechanical systems

Modeling of Translational Mechanical Systems

Translational mechanical systems move along a **straight line**. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper.

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero. Let us now see the force opposed by these three elements individually.

Mass

Mass is the property of a body, which stores **kinetic energy**. If a force is applied on a body having mass **M**, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and frictions are negligible.

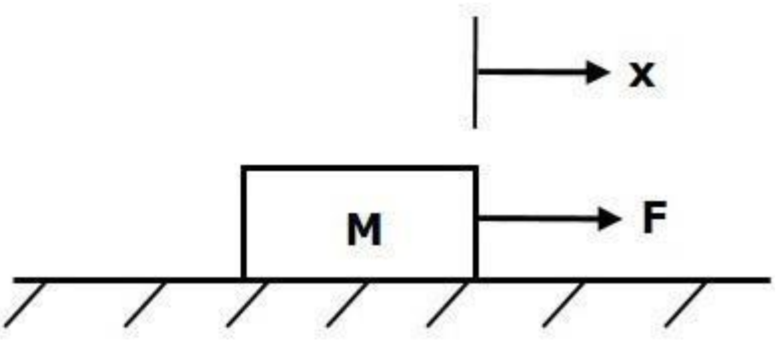

Figure 1.3.4: block diagram of mass

[Source: "Control System Engineering" by Nagoor Kani, page: 1.7]

$F_m$aa

$$\Rightarrow F_m = Ma = M\frac{d^2x}{dt^2}$$

$$F = F_m = Ma = M\frac{d^2x}{dt^2}$$

Where,

- **F** is the applied force
- **F_m** is the opposing force due to mass
- **M** is mass
- **a** is acceleration
- **x** is displacement

Spring

Spring is an element, which stores **potential energy**. If a force is applied on spring **K**, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction are negligible.

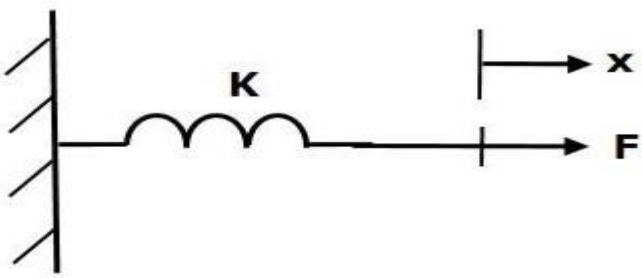

Figure 1.3.5: block diagram of spring

[Source: "Control System Engineering" by Nagoor Kani, page: 1.7]

$$F \alpha x$$

$$\Rightarrow F_k = Kx$$

$$F = F_k = Kx$$

Where,

- **F** is the applied force
- $\mathbf{F_k}$ is the opposing force due to elasticity of spring
- **K** is spring constant
- **x** is displacement

Dashpot

If a force is applied on dashpot **B**, then it is opposed by an opposing force due to **friction** of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity are negligible.

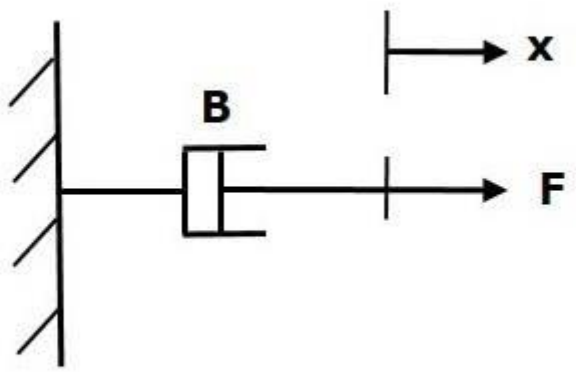

Figure 1.3.6: block diagram of friction

[Source: "Control System Engineering" by Nagoor Kani, page: 1.7]

$$F_b \ \alpha \ \nu$$

$$\Rightarrow F_b = B\nu = B\ dx/dt$$

$$F = F_b = B\ dx/dt$$

Where,

- $\mathbf{F_b}$ is the opposing force due to friction of dashpot
- **B** is the frictional coefficient

- **v** is velocity
- **x** is displacement

Modeling of Rotational Mechanical Systems

Rotational mechanical systems move about a fixed axis. These systems mainly consist of three basic elements. Those are **moment of inertia, torsional spring** and **dashpot**.

If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero. Let us now see the torque opposed by these three elements individually.

Moment of Inertia

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores **kinetic energy**.

If a torque is applied on a body having moment of inertia **J**, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.

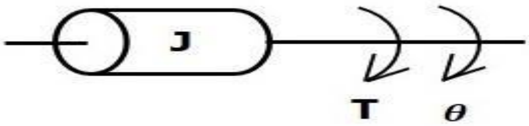

Figure 1.3.7: block diagram of moment of inertia

[Source: "Control System Engineering" by Nagoor Kani, page: 1.15]

Tjaα

$$\Rightarrow T_j = J\alpha = J\, d^2\theta/dt^2$$

$$T = T_j = J\, d^2\theta \,/\, dt^2$$

Where,

- **T** is the applied torque
- **T_j** is the opposing torque due to moment of inertia
- **J** is moment of inertia
- **α** is angular acceleration

- **θ** is angular displacement

Torsional spring

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores **potential energy**.

If a torque is applied on torsional spring **K**, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.

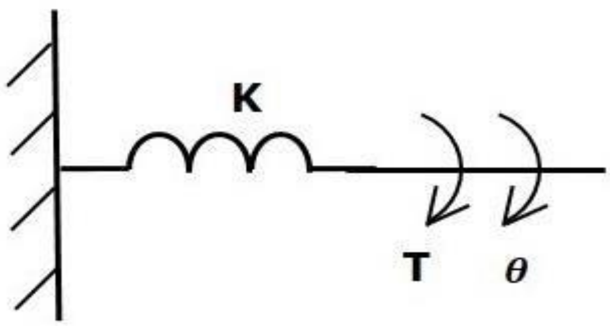

Figure 1.3.8: block diagram of torsional spring

[Source: "Control System Engineering" by Nagoor Kani, page: 1.15]

$$T_k \propto \theta$$

$$\Rightarrow T_k = K\,\theta$$

$$T = T_k = K\,\theta$$

Where,

- **T** is the applied torque
- $\mathbf{T_k}$ is the opposing torque due to elasticity of torsional spring
- **K** is the torsional spring constant
- **θ** is angular displacement

Dashpot

If a torque is applied on dashpot **B**, then it is opposed by an opposing torque due to the **rotational friction** of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.

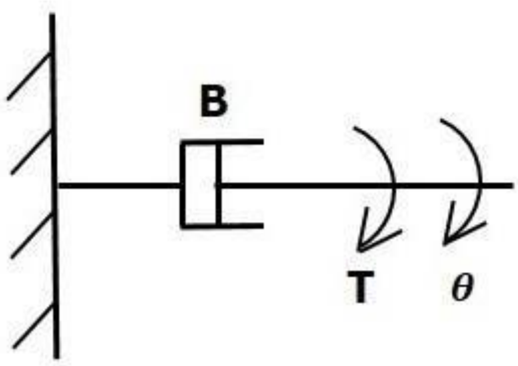

Figure 1.3.9: block diagram of dashpot

[Source: "Control System Engineering" by Nagoor Kani, page: 1.15]

$$T_b \text{ a } \omega$$

$$\Rightarrow T_b = B\omega = B\, d\theta/dt$$

$$T = T_b = B\, d\theta/dt$$

Where,

- $\mathbf{T_b}$ is the opposing torque due to the rotational friction of the dashpot
- **B** is the rotational friction coefficient
- **ω** is the angular velocity
- **θ** is the angular displacement

Electrical Analogies of Mechanical Systems

Two systems are said to be **analogous** to each other if the following two conditions are satisfied.

- The two systems are physically different
- Differential equation modeling of these two systems are same

Electrical systems and mechanical systems are two physically different systems. There are two types of electrical analogies of translational mechanical systems. Those are force voltage analogy and force current analogy.

Force Voltage Analogy

In force voltage analogy, the mathematical equations of **translational mechanical system** are compared with mesh equations of the electrical system.

Consider the following translational mechanical system as shown in the following figure 1.3.10.

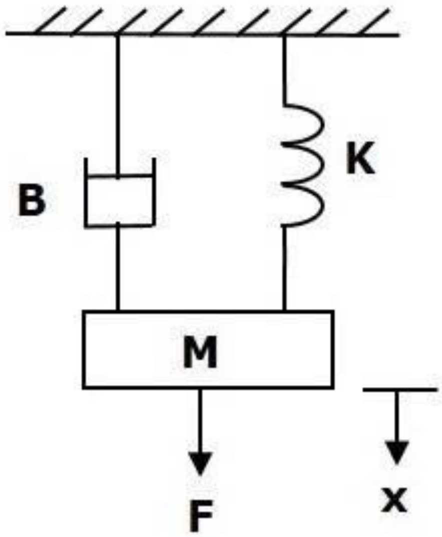

Figure 1.3.10: block diagram mechanical system

[Source: "Control System Engineering" by Nagoor Kani, page: 1.8]

The **force balanced equation** for this system is

$$F=F_m+F_b+F_k$$

$$\Rightarrow F=M\ d^2x/dt^2+B\ dx/dt+Kx$$ **(Equation 1)**

Consider the following electrical system as shown in the following figure 1.3.11. This circuit consists of a resistor, an inductor and a capacitor. All these electrical elements are connected in a series. The input voltage applied to this circuit is V volts and the current flowing through the circuit is I Amps.

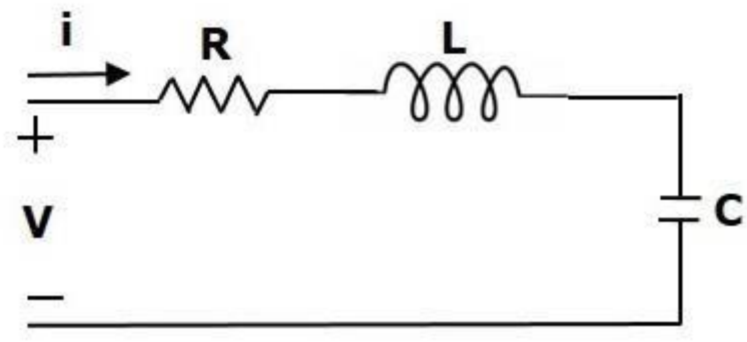

Figure 1.3.11: block diagram of electrical analogy of mechanical system

[Source: "Control System Engineering" by Nagoor Kani, page: 1.8]

Mesh equation for this circuit is

$$V=R_i+L\ di/dt+1/c \int I\ dt$$ **(Equation 2)**

Substitute, i=dq/dt in Equation 2.

$$V=R\ dq/dt+L\ d^2q/dt^2+q/C$$

$$\Rightarrow V=L\ d^2q/dt^2 +R\ dq/dt +(1/c)q \quad \textbf{(Equation 3)}$$

By comparing Equation 1 and Equation 3, we will get the analogous quantities of the translational mechanical system and electrical system. The following table shows these analogous quantities.

Translational Mechanical System	Electrical System
Force(F)	Voltage(V)
Mass(M)	Inductance(L)
Frictional Coefficient(B)	Resistance(R)
Spring Constant(K)	Reciprocal of Capacitance (1/c)
Displacement(x)	Charge(q)
Velocity(v)	Current(i)

Similarly, there is torque voltage analogy for rotational mechanical systems. Let us now discuss about this analogy.

Torque Voltage Analogy

In this analogy, the mathematical equations of **rotational mechanical system** are compared with mesh equations of the electrical system.

Rotational mechanical system is shown in the following figure 1.3.12.

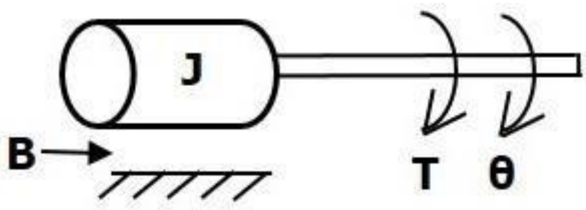

Figure 1.3.12: block diagram of mechanical rotational system

[Source: "Control System Engineering" by Nagoor Kani, page: 1.16]

The torque balanced equation is

$$T=T_j+T_b+T_k$$

$$\Rightarrow T=J\, d^2\theta/dt^2+ B\, d\theta/dt +k\theta \quad \textbf{(Equation 4)}$$

By comparing Equation 4 and Equation 3, we will get the analogous quantities of rotational mechanical system and electrical system. The following table shows these analogous quantities.

Rotational Mechanical System	Electrical System
Torque(T)	Voltage(V)
Moment of Inertia(J)	Inductance(L)
Rotational friction coefficient(B)	Resistance(R)
Torsional spring constant(K)	Reciprocal of Capacitance (1/c)
Angular Displacement(θ)	Charge(q)
Angular Velocity(ω)	Current(i)

Force Current Analogy

In force current analogy, the mathematical equations of the **translational mechanical system** are compared with the nodal equations of the electrical system.

Consider the following electrical system as shown in the following figure 1.3.13. This circuit consists of current source, resistor, inductor and capacitor. All these electrical elements are connected in parallel.

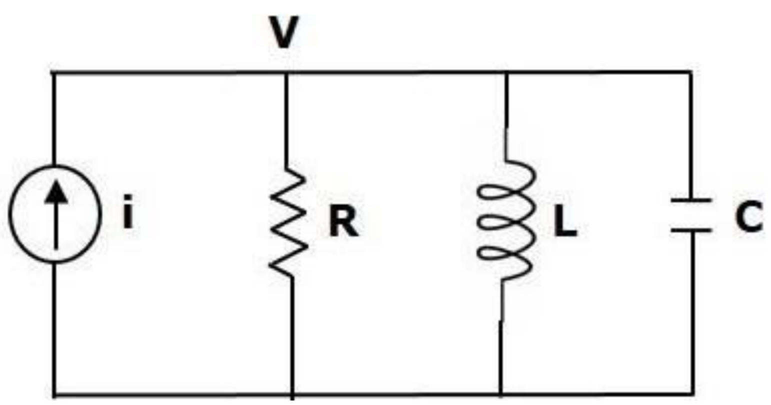

Figure 1.3.13: electrical analogy of mechanical rotational system

[Source: "Control System Engineering" by Nagoor Kani, page: 1.16]

The nodal equation is

$$i = V/R + 1/L \int V dt + C\, dV/dt \quad \textbf{(Equation 5)}$$

Substitute, $V = d\Psi/dt$ in Equation 5.

$$i = 1/R\, d\Psi/dt + (1/L)\Psi + C\, d^2\Psi/dt^2$$

$$\Rightarrow i = C\, d^2\Psi/dt^2 + (1/R)\, d\Psi/dt + (1/L)\Psi \quad \textbf{(Equation 6)}$$

By comparing Equation 1 and Equation 6, we will get the analogous quantities of the translational mechanical system and electrical system. The following table shows these analogous quantities.

Similarly, there is a torque current analogy for rotational mechanical systems. Let us now discuss this analogy.

Torque Current Analogy

Translational Mechanical System	Electrical System
Force(F)	Current(i)
Mass(M)	Capacitance(C)
Frictional coefficient(B)	Reciprocal of Resistance(1/R)
Spring constant(K)	Reciprocal of Inductance(1/L)
Displacement(x)	Magnetic Flux(ψ)
Velocity(v)	Voltage(V)

In this analogy, the mathematical equations of the **rotational mechanical system** are compared with the nodal mesh equations of the electrical system.

By comparing Equation 4 and Equation 6, we will get the analogous quantities of rotational mechanical system and electrical system. The following table shows these analogous quantities.

Rotational Mechanical System	Electrical System

Torque(T)	Current(i)
Moment of inertia(J)	Capacitance(C)
Rotational friction coefficient(B)	Reciprocal of Resistance(1/R)
Torsional spring constant(K)	Reciprocal of Inductance(1/L)
Angular displacement(θ)	Magnetic flux(ψ)
Angular velocity(ω)	Voltage(V)

In this chapter, we discussed the electrical analogies of the mechanical systems. These analogies are helpful to study and analyze the non-electrical system like mechanical system from analogous electrical system.