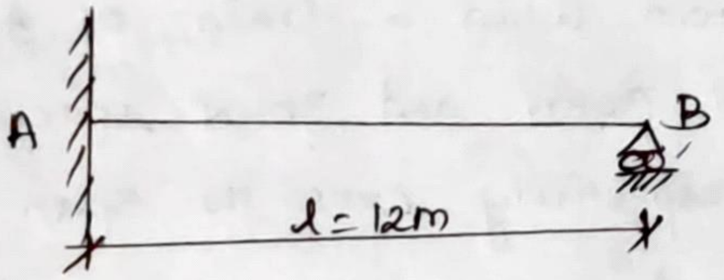
Unit-2: Influence Lines for Indeterminate Structures

Muller-Breslau Principle:

Muller-Breslau principle states that, ~~we~~ The influence line for an assigned function of a statically determinate beam may be obtained by remaing the restraint offered by that function and introducing a directly related generalised unit displacement at the location and in the direction of the function.

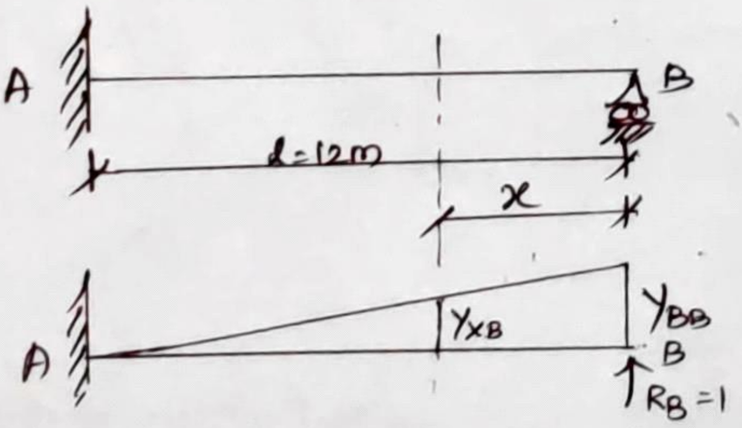
1. Draw the ILD for reaction at B and for the support moment M_A at A for the Cantilever shown in figure. Compute the ILD ordinates at 1.5m intervals



Soln:

(i) Remove the restraint due to R_B.

(ii) Apply a unit displacement (upward) at B.



When $R_B = 1$, then Y_{XB} is displacement at the section due to unit load at B.

$$M_x = -EI\frac{d^2y}{dx^2}$$

$$M_x = R_B \times x \qquad [R_B = 1]$$

$$M_x = x$$

$$EI\frac{d^2y}{dx^2} = -x$$

$$EI\frac{dy}{dx} = -\frac{x^2}{2} + C_1 \rightarrow ①$$

$$EI \cancel{\frac{d^2y}{dx^2}}\, y = -\frac{x^3}{6} + C_1x + C_2 \rightarrow ②$$

$x = 12m$, $y = 0$ $\frac{dy}{dx} = 0$ sub in ① ⇒

$$0 = -\frac{12^2}{2} + C_1$$

$$C_1 = \frac{144}{2} = 72$$

Sub in ② ⇒

$$0 = -\frac{12^3}{6} + 72(12) + C_2$$

$$C_2 = -576$$

$$Y_{xB} = \frac{1}{EI}\left[-\frac{x^3}{6} + C_1x + C_2\right]$$

$$Y_{xB} = \frac{1}{EI}\left[-\frac{x^3}{6} + 72x - 576\right]$$

At $x = 0$, ⇒ Y_{BB}

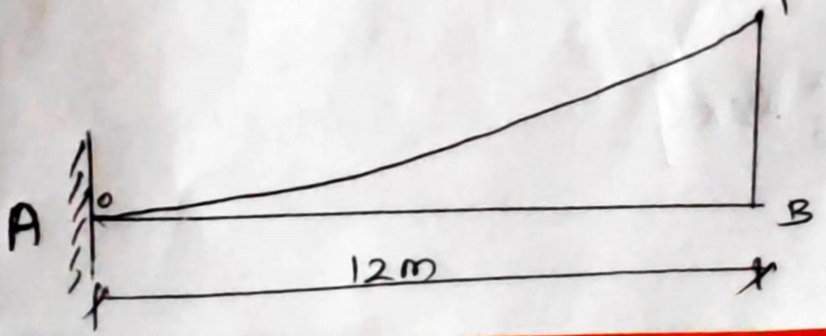
$$Y_{BB} = \frac{-576}{EI}$$

ILD for R_B at $x = \frac{Y_{xB}}{Y_{BB}} = \frac{\frac{1}{EI}\left[\frac{-x^3}{6} + 72x - 576\right]}{\frac{1}{EI}(-576)}$

$$= \frac{-\frac{x^3}{6} + 72x - 576}{-576}$$

Plot ordinates of ILD for R_B at 1.5m intervals

x_m	0	1.5	3	4.5	6	7.5	9	10.5	12
R_B	1	0.814	0.632	0.463	0.312	0.184	0.085	0.022	0

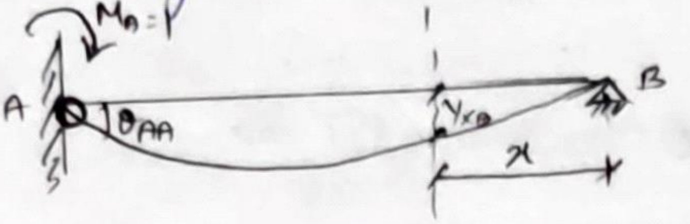


(ii) ILD for M_A:

To get IL for M_A, we have to

(i) Introduce a hinge at A and

(ii) Apply unit rotation at A.



$M_A = 1$ $\qquad R_A + R_B = 0$

$R_A = -R_B$

Taking moment at A,

$R_B \times 12 = 1$

$R_B = 1/12$

$R_A + R_B = 0$

$R_B = -R_A \qquad R_A = -1/12$

$M_x = -EI \dfrac{d^2y}{dx^2} \rightarrow ①$

$M_x = x/12$

① $\Rightarrow$

$EI \dfrac{d^2y}{dx^2} = -x/12$

$EI \dfrac{dy}{dx} = -\dfrac{x^2}{24} + C_1 \rightarrow ②$

$EI\,y = -\dfrac{x^3}{72} + C_1 x + C_2 \rightarrow ③$

At $x = 0$, $y = 0$,

$x = 12$, $y = 0$

③ $\Rightarrow$ $0 = -0/72 + C_1 \times 0 + C_2$

$C_2 = 0$

③ $\Rightarrow$ $0 = -\dfrac{12^3}{72} + 12\,C_1$

$C_1 = 2$

$Y_{xA} = \dfrac{1}{EI}\left[-\dfrac{x^3}{72} + 2x.\right]$

② → $\theta_{AA} = \frac{dy}{dx} = \frac{1}{EI}\left[-\left(\frac{x^2}{24}\right)+2\right]$

At $x = 12$ m,

$$\theta_{AA} = \frac{1}{EI}\left[-\left(\frac{12^2}{24}\right)+2\right]$$

$$\theta_{AA} = -4/EI$$

ILD for $M_A = \frac{y_{xA}}{\theta_{AA}} = \frac{\frac{1}{EI}\left[-\frac{x^3}{72}+2x\right]}{\frac{1}{EI}(-4)}$

$$= \frac{-x^3/72 + 2x}{-4}$$

$$= -\frac{x^3}{72\times -4} + \frac{2x}{-4}$$

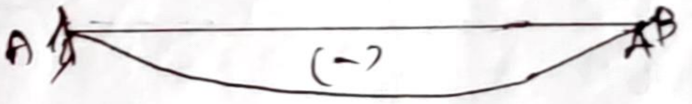
ILD for $M_A = \frac{x^3}{288} - x/2$

To find ILD for M_A at 1.5m intervals

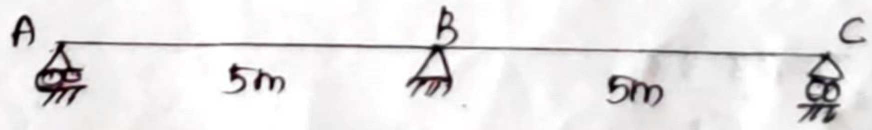
$x = 0$, ILD $= 0$

$x = 1.5$ ILD $= -0.738$

x	0	1.5	3	4.5	6	7.5	9	10.5	12
ILD	0	-0.738	-1.406	-1.934	-2.250	-2.285	-1.969	-1.230	0



2. Determine the influence line for R_A for the continuous beam shown in figure. Compute the IL ordinates at 1m intervals.



Soln :

$R_A + R_B + R_C = 0$

Apply unit force at A $\Sigma M_C = 0$

$R_A \times 10 + R_B \times 5 = 0$

$1 \times 10 + R_B \times 5 = 0$

$10 + 5R_B = 0$

$R_B = -2$kN

$\Sigma M_A = 0$

$(R_C \times 10) + (R_B \times 5) = 0$

$10 R_C - (2 \times 5) = 0$

$10 R_C = 10$

$R_C = 1\ kN$

$$M_x = -EI \frac{d^2y}{dx^2} \longrightarrow Ⓐ$$

Consider a section x from C in the span ABC

$$M_x = 1 \times x \,\vdots\, - 2(x-5)$$

$$EI \frac{d^2y}{dx^2} = -x + \vdots\, 2(x-5) \rightarrow ①$$

$$EI \frac{dy}{dx} = -\frac{x^2}{2} + C_1 + \vdots\, \frac{2(x-5)^2}{2} \longrightarrow ②$$

$$EIy = -\frac{x^3}{6} + C_1 x + C_2 + \vdots\, \frac{2(x-5)^3}{6} \rightarrow ③$$

At $x = 0$, $y = 0$

$③ \Rightarrow 0 = C_2$

At $x = 5$, $y = 0$,

$$③ \Rightarrow EI(0) = -\frac{5^3}{6} + C_1 5 + 0$$

$$C_1 = 4.167$$

$$y_{xA} = \frac{1}{EI}\left[-\frac{x^3}{6} + 4.167x \,\vdots\, + \frac{(x-5)^3}{3}\right]$$

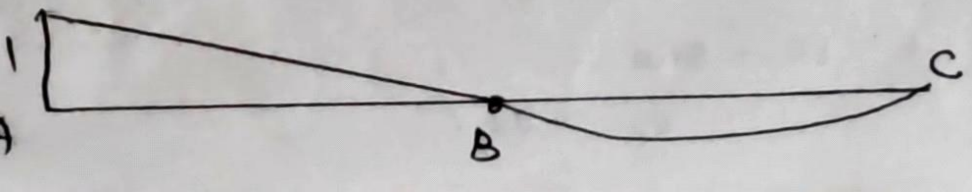
At $x = 10\,m$

$$y_{AA} = \frac{1}{EI}\left[-\frac{10^3}{6} + (4.167 \times 10) + \frac{(10-5)^3}{3}\right]$$

$$= \frac{-83.33}{EI}$$

$$ILD \text{ at } R_A = \frac{y_{xA}}{y_{AA}} = \frac{-x^3/6 + 4.167x + \frac{(x-5)^3}{3}}{-83.33}$$

x	0	1	2	3	4	5	6	7	8	9	10
R_A	0	−0.048	−0.084	−0.096	−0.072	0	0.128	0.304	0.516	0.752	1



3. Determine the influence line for the reaction at the middle support B of continuous beam as shown in figure. Compute the ordinates at every 1m interval.

Soln:

Remove support at B ~~at~~ and apply a unit force at B.

$R_B = 1$, $R_A + R_B + R_C = 0$

$\Sigma M_A = 0.$

$R_C \times 10 + R_B \times 5 = 0$

$R_C \times 10 + 1 \times 5 = 0$

$10 R_C = -5$

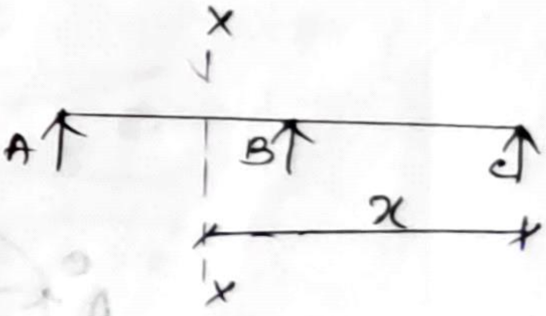
$R_C = -5/10 = -0.5$

$\Sigma V = 0$

$R_A + 1 - 0.5 = 0$

$R_A = -0.5$

Consider a section xx from C,



$M_x = -EI \dfrac{d^2y}{dx^2} \rightarrow ①$

$M_x = -0.5x + \vdots\, 1(x-5)$

$① \Rightarrow EI \dfrac{d^2y}{dx^2} = 0.5x - \vdots\, 1(x-5)$

$EI \dfrac{dy}{dx} = \dfrac{0.5x^2}{2} + C_1 \;\vdots\; - \dfrac{(x-5)^2}{2}$

$EI\, y = \dfrac{0.5x^3}{6} + C_1 x + C_2 \;\vdots\; - \dfrac{(x-5)^3}{6} \rightarrow ②$

When $x = 0$, $y = 0$,

$② \Rightarrow EI(0) = 0 + C_2$

$C_2 = 0$

When $x = 10$, $y = 0$

$$② \rightarrow 0 = \frac{0.5 \times 10^3}{6} + C_1 \times 10 + 0 - \frac{(10-5)^3}{6}$$

$$C_1 = -6.244$$

$$Y_{xB} = \frac{1}{EI}\left[\frac{0.5x^3}{6} - 6.249x \;\vdots - \frac{(x-5)^3}{6}\right]$$

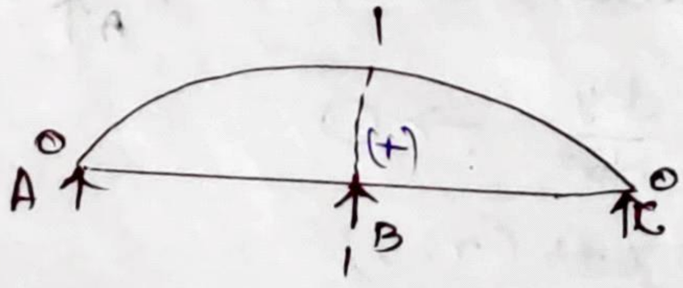
At $x = 5m$

$$Y_{BB} = \frac{1}{EI}\left[\frac{0.5(5)^3}{6} - (6.249 \times 5)\right]$$

$$= -\frac{20.828}{EI}$$

ILD for $R_B = \frac{Y_{xB}}{Y_{BB}} = \frac{\frac{1}{EI}\left[\frac{0.5x^3}{6} - 6.249x \;\vdots - \frac{(x-5)^3}{6}\right]}{-\frac{20.828}{EI}}$

$$= \frac{\left[\frac{0.5x^3}{6} - 6.249x \;\vdots - \left(\frac{x-5}{6}\right)^3\right]}{-20.828}$$



ILD for R_B

x	(C) 0	1	2	3	4	(B) 5	6	7	8	9	(A) 10
R_B	0	0.296	0.568	0.792	0.044	1	0.943	0.791	0.567	0.295	0

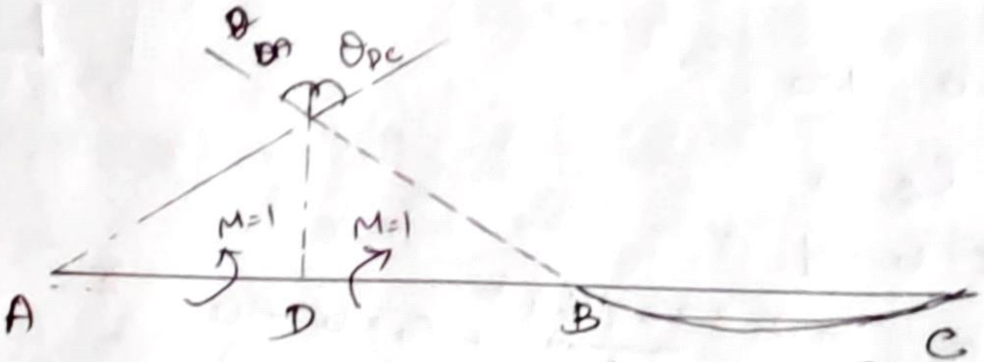
4. Using Muller Breslau's principle, draw the influence line for bending moment at the mid point D of span AB of the continuous beam ABC as shown in figure. Determine the influence line ordinates at suitable intervals and plot them.

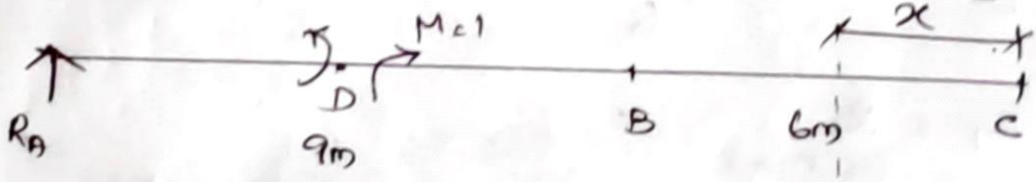
Soln:

(i) Introduce a hinge at D.

(ii) Apply unit BM at D.

(iii) Determine the deflection Y_{xD} and x and the slope θ_{DD} at D. $\frac{Y_{xD}}{\theta_{DD}}$ is the influence line ordinate at any x.





M = 1 at D,

$R_A \times 4.5 = 1$

$R_A = 1/4.5 = 0.222$

$R_A + R_{D_1} = 0$

$0.222 + R_{D_1} = 0$

$R_{D_1} = -0.222$

$R_{D_2} = 0.222$

$\Sigma M_c = 0.$

$(R_{D_2} \times 10.5) + 1 + (R_B \times 6) = 0$

$(0.222 \times 10.5) + 1 + R_B \times 6 = 0$

$R_B = -0.555$

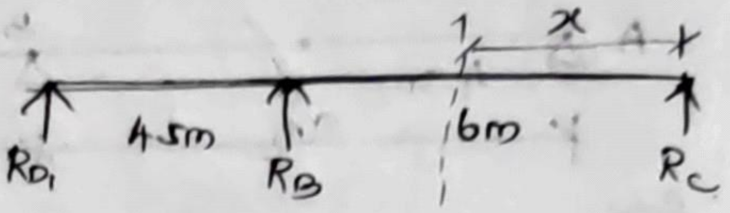
$R_{D_2} + R_B + R_C = 0$

$0.222 - 0.555 + R_C = 0$

$R_C = 0.333$

The two regions AD and DBC will be considered separately (because discontinuity at D)

$M_x = -EI \frac{d^2y}{dx^2} \rightarrow ①$



$M_x = 0.333x \;\vdots\; -0.555(x-6)$

$$EI \frac{d^2y}{dx^2} = -0.333x + \vdots\; 0.555(x-6)$$

$$EI \frac{dy}{dx} = -\frac{0.333x^2}{2} + c_1 + \vdots\; 0.555 \frac{(x-6)^2}{2} \rightarrow ②$$

$$EIy = -\frac{0.333x^3}{6} + c_1x + C_2 + \vdots\; 0.555 \frac{(x-6)^3}{2}$$

$\rightarrow ③$

When $x = 0,\ y = 0$

$\Rightarrow ③ \quad C_2 = 0$

When $x = 6,\ y = 0.$

$$0 = \frac{-0.333(6)^3}{6} + c_1 \times 6 + 0$$

$c_1 = 2$

$$EI \frac{dy}{dx} = \frac{-0.333x^2}{2} + 2 \;\vdots + \frac{0.555(x-6)^2}{2} \rightarrow ④$$

at $x = 10.5$m (at D)

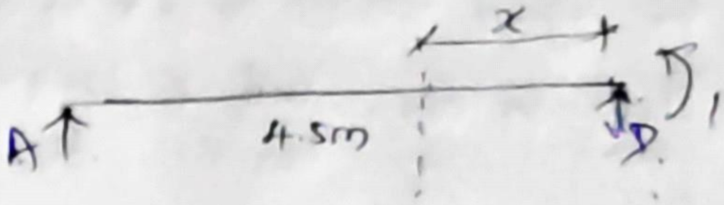
$$\theta_{DC} = \frac{1}{EI}\Big[\frac{-0.333 \times 10.5^2}{2} + 2 + \vdots\; \frac{0.555(10.5-6)^2}{2}\Big)$$

$$= \frac{1}{EI}(-10.736)$$

$$Y_D = \frac{1}{EI}\Big[\frac{-0.333 \times 10.5^3}{6} + 2 \times 10.5) + \frac{0.555(10.5-6)^3}{6}$$

$$= \frac{-34.82}{EI}$$

For Zone AD:



$M_x = -EI \frac{d^2y}{dx^2}$

$M_x = -0.222x + 1$

$EI \frac{d^2y}{dx^2} = 0.222x - 1 \rightarrow ⑤$

$EI \frac{dy}{dx} = \frac{0.222x^2}{2} - x + C_3 \rightarrow ⑥$

$EI \cancel{\frac{dy}{dx}} Y = \frac{0.222x^3}{6} - \frac{x^2}{2} + C_3 x + C_4 \rightarrow ⑦$

At $x = 0$, $Y = \frac{-34.82}{EI}$

$⑦ \Rightarrow \cancel{EI}\left(\frac{-34.82}{EI}\right) = \frac{0.222(0)}{6} - \frac{0}{2} + C_3 \times 0 + C_4$

$C_4 = -34.82$

At $x = 4.5$, $Y = 0$

$⑦ \Rightarrow 0 = \frac{0.222 \times 4.5^3}{6} - \frac{4.5^2}{2} + 4.5C_3 - 34.52$

$C_3 = 9.24$

$⑥ \Rightarrow EI \frac{dy}{dx} = \frac{0.222x^2}{2} - x + 9.24$

$\theta_{DA} = \left(\frac{dy}{dx}\right)_{x=0} = \left(\frac{0.222(0)^2}{2} - 0 + 9.24\right) 1/EI$

$\theta_{DA} = \frac{9.24}{EI} \rightarrow ⑧$

$EI\,Y = \frac{0.222x^3}{6} - \frac{x^2}{2} + C_3 x + C_4$

$= 1/EI \left[\frac{0.222x^3}{6} - \frac{x^2}{2} + 9.238 - 34.82\right] \rightarrow ⑦$

$\theta_{DD} = \theta_{DA} - \theta_{DC}$

$= \frac{9.24}{EI} + \frac{10.738}{EI} = \frac{19.978}{EI}$

For region CD (0 to 10.5 ⇒ C to D)

$$\text{ILD for } M_D = \frac{y_{xD}}{\theta_{DD}} = \frac{-0.333\frac{x^3}{6} + 2x}{19.978} + \frac{0.555(x-6)^2}{2}$$

For region DA (10.5 to 15) ⇒ D to A

$$\text{ILD for } M_D = \left[\frac{0.222\frac{x^3}{6} - \frac{x^2}{2} + 9.24x - 34.82}{19.978}\right]$$

$x = 3m$

$$M_3 = \frac{-0.333\frac{3^3}{6} + 2(3)}{19.978} = 0.225$$

$x = 9m$

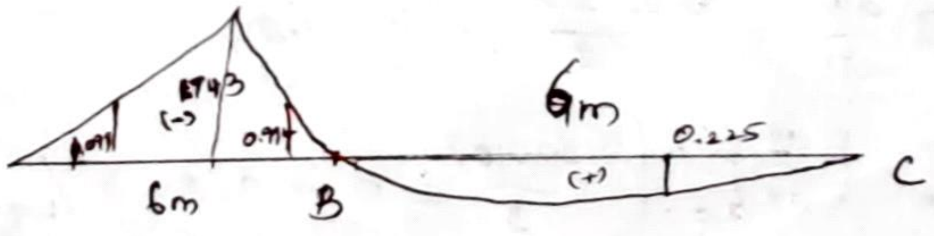
$$M_9 = \frac{-0.333 \times \frac{9^3}{6} + 2(9) + 0.555\frac{(9-6)^3}{6}}{19.978} = -0.999$$

$(x = 12 - 10.5 = 1.5)$

$x = 12m$

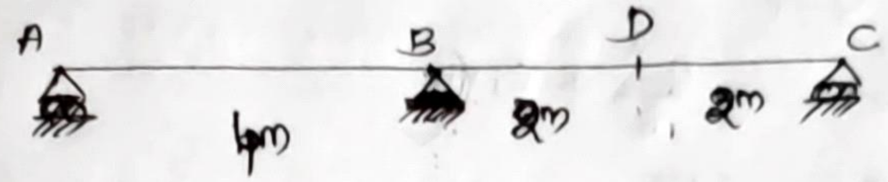
$$M_{12} = \frac{0.222 \times \frac{1.5^3}{6} - \frac{1.5^2}{2} + (9.24 \times 1.5) - 34.82}{19.978}$$

$M_{12} = -1.099$

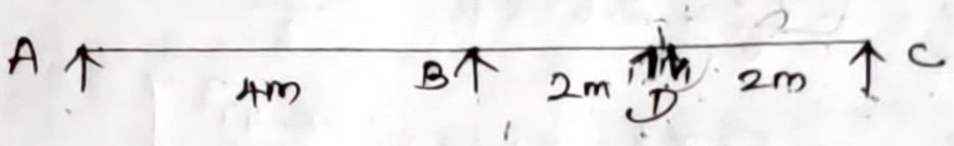


x from C	0	3	6	9	10.5	12	15
ILD	0	0.225	0	-0.999	-1.743	-1.099	0.

5. Determine the influence line for shear force at D, the mid point of span BC of a continuous beam shown in fig. Compute the ILD at 1.0m intervals.



Soln:



$\Sigma V = 0$ for DC, $\qquad R_{D_2} = -1$

$R_{D_2} + R_C = 0 \qquad M_D = 1 \times 2 = 2$ kNm.

$-1 + R_C = 0$

$R_C = 1$

$\Sigma M_A = 0$ for ABD, $\qquad R_{D_1} = 1$

$R_{D_1} \times 6 + R_B \times 4 + 2 = 0$

$1 \times 6 + R_B \times 4 + 2$

$R_B \times 4 = -8$

$R_B = -8/4$

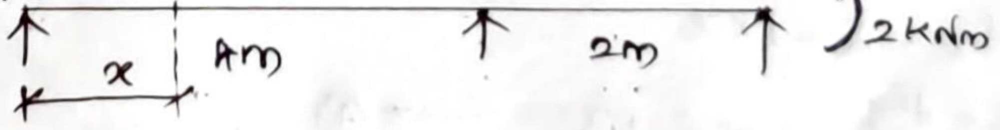
$R_B = -2$

$R_A + R_B + R_{D_1} = 0$

$R_A - 2 + 1 = 0$

$R_A = 1$

Portion ABD:



$-M_x = EI \dfrac{d^2y}{dx^2}$

$M_x = 1 \times x - 2(x-4)$

$EI \dfrac{d^2y}{dx^2} = -x + 2(x-4) \longrightarrow ①$

$EI \dfrac{dy}{dx} = -\dfrac{x^2}{2} + \dfrac{2(x-4)^2}{2} + C_1 \longrightarrow ②$

$EI\, y = -x^3/6 + C_1 x + C_2 + \dfrac{(x-4)^3}{3} \longrightarrow ③$

At $x=0,\ y=0$ slope !

③ $\Rightarrow EI(0) = -0^3/6 + C_1 \times 0 + C_2$

$C_2 = 0$

At $x = 4m,\ y = 0$

⑧ $\Rightarrow EI(0) = -\frac{4^3}{6} + C_1 \times 4$

$C_1 = 2.67$

③ $\Rightarrow EIy_x = -\frac{x^3}{6} + 2.67x + \vdots \frac{(x-4)^3}{3}$

$$y_x = \frac{1}{EI}\left[-\frac{x^3}{6} + 2.67x + \vdots \frac{(x-4)^3}{3}\right]$$

$x = 6$

$$y_D = \frac{1}{EI}\left[-\frac{6^3}{6} + 2.67 \times 6 + \vdots \frac{(6-4)^3}{3}\right]$$

$$= \frac{1}{EI}\left[-36 + 16.02 + \vdots 2.667\right]$$

$$= -\frac{17.313}{EI}$$

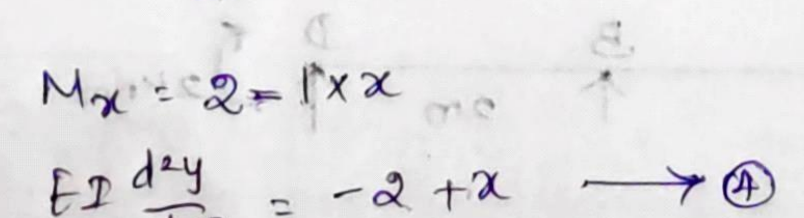
put $x = 6$ in ②

$$\theta_D = \frac{1}{EI}\left(-\frac{6^2}{2} + \vdots \frac{(6-4)^2}{2} + 2.67\right)$$

$$= \frac{1}{EI}\left(-18 + 4 + 2.67\right)$$

$$= -\frac{11.33}{EI}$$

Portion DC :



$$EI\frac{d^2y}{dx^2} = -M_x$$

$M_x = 2 - 1 \times x$

$EI\frac{d^2y}{dx^2} = -2 + x \longrightarrow$ ④

$EI\frac{dy}{dx} = -2x + \frac{x^2}{2} + C_1 \longrightarrow$ ⑤

$EIy = -\frac{2x^2}{2} + \frac{x^3}{6} + C_1 x + C_2 \longrightarrow$ ⑥

At $x=0,\ y = -\frac{17.313}{EI}$

At $x=0,\ \theta = \frac{-11.33}{EI}$

$$\cancel{⑥ \Rightarrow \cancel{EI}\left(\frac{-17.313}{\cancel{EI}}\right) = 0 + C_2}$$

$$\cancel{C_2 = -17.313}$$

$$⑤ \Rightarrow \cancel{EI}\left(-\frac{11.33}{\cancel{EI}}\right) = -2(0) + 0^2/2 + C_1$$

$$C_1 = -11.33$$

$$⑥ \Rightarrow EI\,y = -\frac{\cancel{2}x^2}{\cancel{2}} + x^3/6 + (-11.33x) + C_2$$

$x = 2,\ y = 0.$

$$0 = -2^2 + 2^3/6 + (-11.33 \times 2) + C_2$$

$$= -4 + 1.33 - 22.66 + C_2$$

$$0 = -25.33 + C_2$$

$$C_2 = 25.33$$

$$EIy = -x^2 + \frac{x^3}{6} - 11.33x + 25.33$$

$x = 0$

$$y_{DC} = 1/EI\left[0 + 0 - 0 + 25.33\right]$$

$$= 1/EI\,(25.33)$$

$$y_{DD} = y_{DA} - y_{DC}$$

$$= \frac{-17.313}{EI} - \frac{25.33}{EI}$$

$$y_{DD} = \frac{-42.643}{EI}$$

For span ABD,

$$ILD = \frac{y_{XD}}{y_{DD}} = \frac{1/\cancel{EI}\left[-\frac{x^3}{6} + 2.67x + \left|\ \frac{(x-4)^3}{3}\right.\right]}{-42.643/\cancel{EI}}$$

$$= \frac{\frac{x^3}{6} - 2.67x - \frac{(x-4)^3}{3}}{42.643}$$

For span DC,

$$ILD = \frac{y_{XD}}{y_{DD}} = \frac{\left[-x^2 + x^3/6 - 11.33x + 25.33\right] \times 1/\cancel{EI}}{-42.643/\cancel{EI}}$$

$$ILD = \frac{x^2 - x^3/6 + 11.33x - 25.33}{42.643}$$

x	0	1	2	3	4	5	6 (L)	6 (R)	7	8
ILD	0	-0.059	-0.094	-0.082	0	0.168	0.406	-0.594	-0.308	0