



Unit - II

Combinatorics

Principle of Mathematical Induction:

Let $P(n)$ be a statement or proposition involving for all positive integers n .

Step 1: $P(1)$ is true.

Step 2: Assume that $P(k)$ is true

Step 3: we have to prove that $P(k+1)$ is true.

Problems:

1. Prove that $2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$ by mathematical induction.

$$P(n) : 2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$$

$$\begin{aligned} \text{Step 1: } P(1) &= 2^1 = 2^{1+1} - 2 \\ &= 2^2 - 2 \\ &= 4 - 2 \end{aligned}$$

$$2 = 2$$

$\therefore P(1)$ is true.

Step 2: Assume that

$$P(k) : 2^1 + 2^2 + \dots + 2^k = 2^{k+1} - 2 \text{ is true.}$$

Step 3: To prove $P(k+1)$ is true.

Now

$$\begin{aligned} P(k+1) &= 2^1 + 2^2 + \dots + 2^k + 2^{k+1} \\ &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 2 \\ &= 2^{(k+1)+1} - 2 \end{aligned}$$

$\therefore P(k+1)$ is true.

$\therefore P(n) : 2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$ is true, for all n .



Q. prove that $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$

by mathematical Induction

Let $P(n) : 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: $P(1)$:

LHS : $1^2 = 1$

RHS : $\frac{1(1+1)(2+1)}{6} = \frac{2(3)}{6} = 1$

$\therefore P(1)$ is true.

Step 2: Assume that

$P(k) : 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

Step 3: TO prove $P(k+1)$ is true.

Now, $P(k+1) = 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \frac{(k+1) [k(2k+1) + 6(k+1)]}{6}$

$= \frac{(k+1) [2k^2 + k + 6k + 6]}{6}$

$= \frac{(k+1) [2k^2 + 7k + 6]}{6}$

$= \frac{(k+1) [2k^2 + 4k + 3k + 6]}{6}$

$= \frac{(k+1) [2k(2k+2) + 3(k+2)]}{6}$

$= \frac{(k+1) [(k+2)(2k+3)]}{6}$

$= \frac{(k+1)(k+1+1)(2k+2+1)}{6}$



$$= \frac{(k+1) [(k+1)+1] [2(k+1)+1]}{6}$$

$\therefore P(k+1)$ is true.

Hence $P(n): \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ is true, $\forall n$.

3] Prove that $1+2+\dots+n = \frac{n(n+1)}{2}$

Let $P(n): 1+2+\dots+n = \frac{n(n+1)}{2}$

Step 1:

$$P(1): \text{LHS} = 1$$

$$\text{RHS} = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$$

$\therefore P(1)$ is true.

Step 2: Assume that

$P(k): 1+2+\dots+k = \frac{k(k+1)}{2}$ is true

Step 3: To prove $P(k+1)$ is true.

$$\text{Now, } P(k+1) = 1+2+\dots+k+k+1$$

$$= \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1) [(k+1)+1]}{2}$$

$\therefore P(k+1)$ is true.

$\therefore P(n): 1+2+\dots+n = \frac{n(n+1)}{2}$ is true, for all n .

4] Show that $8^n - 3^n$ is a multiple of 5.

Let $P(n): 8^n - 3^n$ be a multiple of 5.

Step 1: $P(1): 8^1 - 3^1 = 5$ is a multiple of 5
 $\therefore P(1)$ is true.

Step 2: Assume that

$P(k): 8^k - 3^k$ is a multiple of 5, is true.

$$\text{i.e., } 8^k - 3^k = 5m \Rightarrow 8^k = 3^k + 5m$$

Step 3: To prove $P(k+1)$ is true.

$$\text{Now } P(k+1) = 8^{k+1} - 3^{k+1}$$

$$= 8^k \cdot 8^1 - 3^k \cdot 3^1$$

$$= (3^k + 5m) \cdot 8 - 3^k \cdot 3$$

$$= 5m \cdot 8 + 3^k \cdot 8 - 3^k \cdot 3$$

$$= 5m \cdot 8 + 3^k(8 - 3)$$

$$= 5m \cdot 8 + 3^k \cdot 5$$

$$= 5 [8m + 3^k] \text{ which is a multiple of 5, } \forall n$$

$\therefore P(k+1)$ is true.

$\therefore P(n): 8^n - 3^n$ is a multiple of 5, $\forall n$

Q]. Use mathematical induction, prove that $3^n + 7^n - 2$ is divisible by 8, for $n \geq 1$

Let $P(n): 3^n + 7^n - 2$ be divisible by 8, for $n \geq 1$

Step 1: $P(1): 3^1 + 7^1 - 2 = 3 + 7 - 2 = 8$ is divisible by 8, $n \geq 1$

Step 2: Assume that

$P(k): 3^k + 7^k - 2$ is divisible by 8, $n \geq 1$ is true.

$$\text{i.e., } P(k) = \frac{3^k + 7^k - 2}{8} = m$$

$$\Rightarrow 3^k + 7^k - 2 = 8m$$

$$\Rightarrow 3^k = 8m + 2 - 7^k$$

Step 3: To prove $P(k+1)$ is true.

NOW $P(k+1) = 3^{k+1} + 7^{k+1} - 2$

$$= 3^k \cdot 3 + 7^k \cdot 7 - 2$$

$$= (8m+2-7^k) \cdot 3 + 7^k \cdot 7 - 2$$

$$= 24m+6 - 3 \cdot 7^k + 7^k \cdot 7 - 2$$

$$= 24m+4 + 7^k(7-3)$$

$$= 24m+4 + 7^k \cdot 4$$

$$= 4(6m+1+7^k), \quad m \geq 1, k \geq 1$$

$\therefore 6m+1+7^k$ is an even number.
 $\Rightarrow 4(6m+1+7^k)$ is divisible by 8.

$\therefore P(k+1)$ is true.

$\therefore P(n) = 3^n + 7^n - 2$ is divisible by 8, $n \geq 1$.

6]. Prove that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let $P(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1: $P(1)$

LHS: $\frac{1}{1 \cdot 2} = \frac{1}{2}$

RHS: $\frac{1}{1+1} = \frac{1}{2}$

Step 2: Assume that

$P(k) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Step 3: To prove $P(k+1)$ is true.

NOW, $P(k+1) = \frac{1}{1 \cdot 2} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1}$$

② $n^3 + 2n$ is divisible by 3

① $n < 2$
 $1 < 2$
 $k < 2k$
 $k+1 < 2k+1$
 $< 2k+2k$