2.4 FIRST AND SECOND ORDER SYSTEM RESPONSE

Transfer Function

- It is the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions.
- One of the types of modeling a system
- Using first principle, differential equation is obtained
- Laplace Transform is applied to the equation assuming zero initial conditions

Order of a system

- ✓ Order of a system is given by the order of the differential equation governing the system
- ✓ Alternatively, order can be obtained from the transfer function
- ✓ In the transfer function, the maximum power of s in the denominator polynomial gives the order of the system

Dynamic Order of Systems

- Order of the system is the order of the differential equation that governs the dynamic behaviour
- Working interpretation: Number of the dynamic elements / capacitances or holdup elements between a manipulated variable and a controlled variable
- Higher order system responses are usually very difficult to resolve from one another
- The response generally becomes sluggish as the order increases

SYSTEM RESPONSE

First-order system time response

- Transient
- Steady-state

Second-order system time response

- Transient
- Steady-state

FIRST ORDER SYSTEM

Response of First Order System for Unit Step Input

The standard form of closed loop transfer function of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+sT}$$

If the input is unit step, then r(t) and R(s)=1/s

$$C(s) = R(s)\,\frac{1}{1+sT} = \frac{1}{s}\times\frac{1}{1+sT}$$

Applying partial fraction expansion,

$$C(s) = \frac{A}{s} + \frac{B}{1+sT}$$

On solving,

$$C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

On taking inverse Laplace transform, the response in time domain is obtained as,

$$c(t) = 1 - e^{-\frac{t}{T}}$$

Hence, the input and output signal of the first order system is given by,

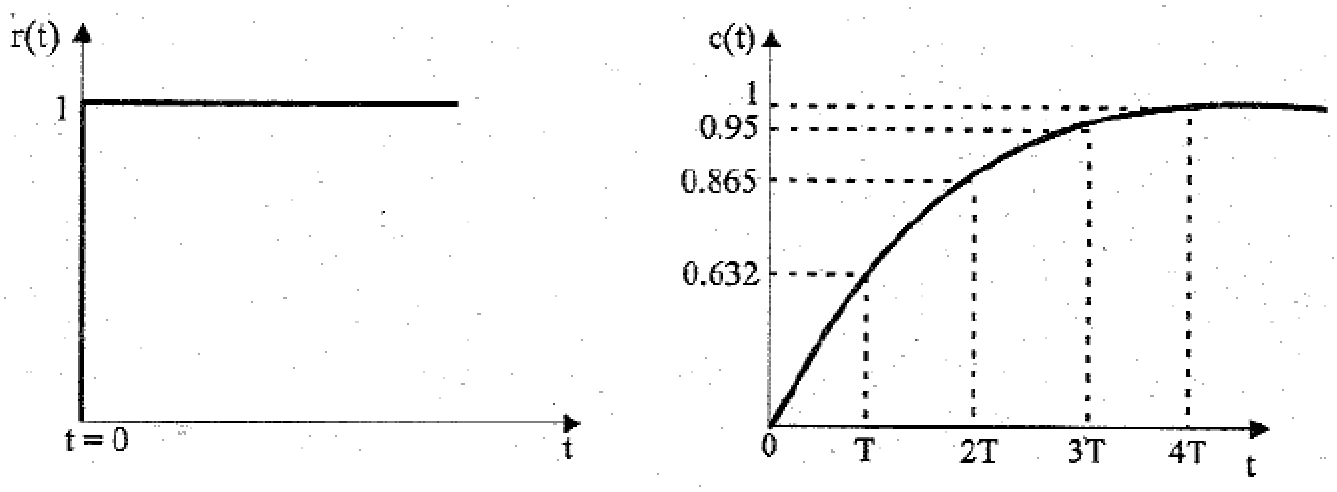

Figure 2.4.1 Response of first order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.20]

SECOND ORDER SYSTEM

LTI second-order system

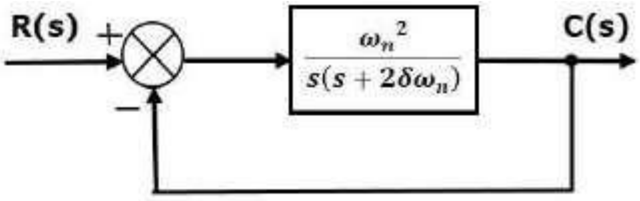

Figure 2.4.2 Closed loop for second order system

[Source: "Control Systems" by Nagoor Kani, Page: 2.20]

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\zeta\omega_n}\right)}{1 + \left(\frac{\omega_n^2}{s(s+2\zeta\omega_n}\right)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ζ is the damping ratio, ω_n is the natural frequency

DAMPING RATIO

It is the ratio of critical damping to actual damping.

CHARACTERISTIC EQUATION

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The roots of characteristic equation are:

- The two roots are imaginary when $\zeta = 0$ (undamped system)
- The two roots are real and equal when $\zeta = 1$ (critically damped system)
- The two roots are real but not equal when $\zeta > 1$ (overdamped system)
- The two roots are complex conjugate when $0 < \zeta < 1$ (underdamped system)

Response of Second Order System for Unit Step Input

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is

$$R(s) = 1/s$$

Transfer function of the second order closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Case 1: Undamped system

When $\zeta = 0$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

For unit step input, R(s) = 1/s,

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Taking inverse Laplace transform,

$$c(t) = 1 - \cos\omega_n t$$

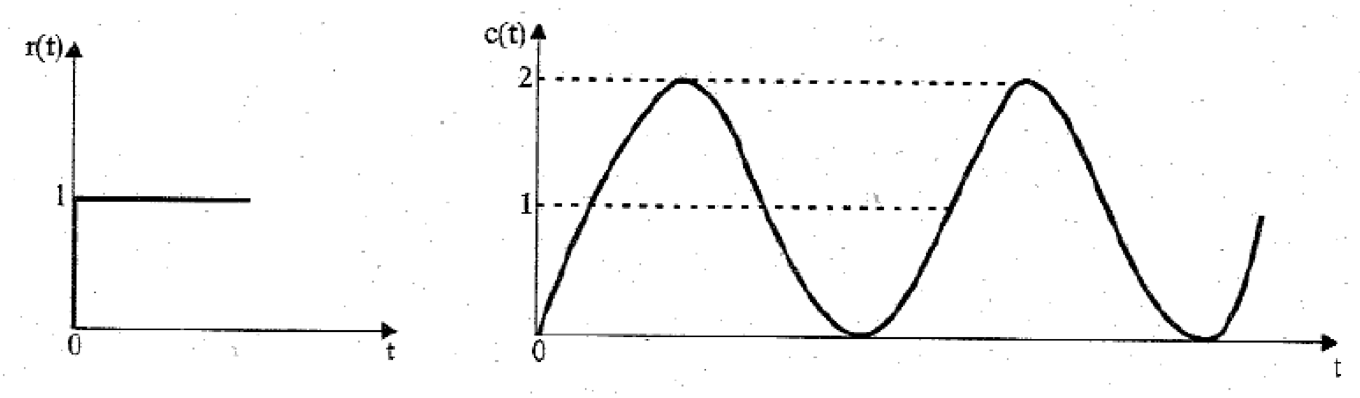

Figure 2.4.3 Response of undamped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.22]

Case 2: Underdamped system

When $0 < \zeta < 1$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = \{s^2 + 2\zeta\omega_n s + (\zeta\omega_n)^2\} + \omega_n^2 - (\zeta\omega_n)^2$$

$$= (s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

For unit step input, R(s)=1/s,

$$C(s) = \frac{\omega_n^2}{s((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

By applying partial fraction,

$$C(s) = \frac{A}{s} + \frac{Bs + C}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

On solving, we get,

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))} - \frac{\zeta\omega_n}{((s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2))}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)} - \frac{\zeta\omega_n}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)}$$

$$- \frac{\zeta}{\sqrt{1 - \zeta^2}} \frac{\omega_n\sqrt{1 - \zeta^2}}{((s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2)}$$

On taking inverse Laplace transform,

$$c(t) = (1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t)$$

$$c(t) = (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}((\sqrt{1-\zeta^2})\cos\omega_d t + \zeta\sin\omega_d t))$$

We know, $\sin\theta = \sqrt{1-\zeta^2}, \cos\theta = \zeta$

$$c(t) = (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t))$$

$$c(t) = (1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin(\omega_d t + \theta)))$$

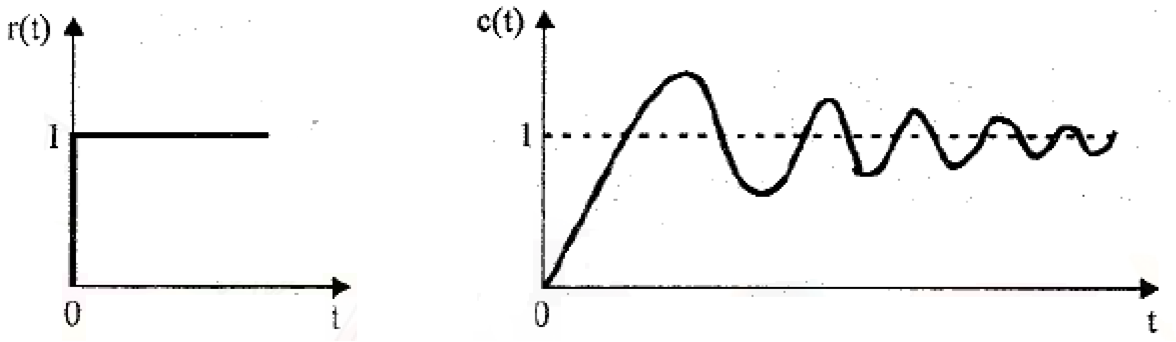

Figure 2.4.4 Response of underdamped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.24]

Case 3: Critically damped system

When $\zeta = 1$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

For a step input, R(s)=1/s

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

By applying partial fractions,

$$C(s) = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

On solving, we get

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

By taking inverse Laplace transform,

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

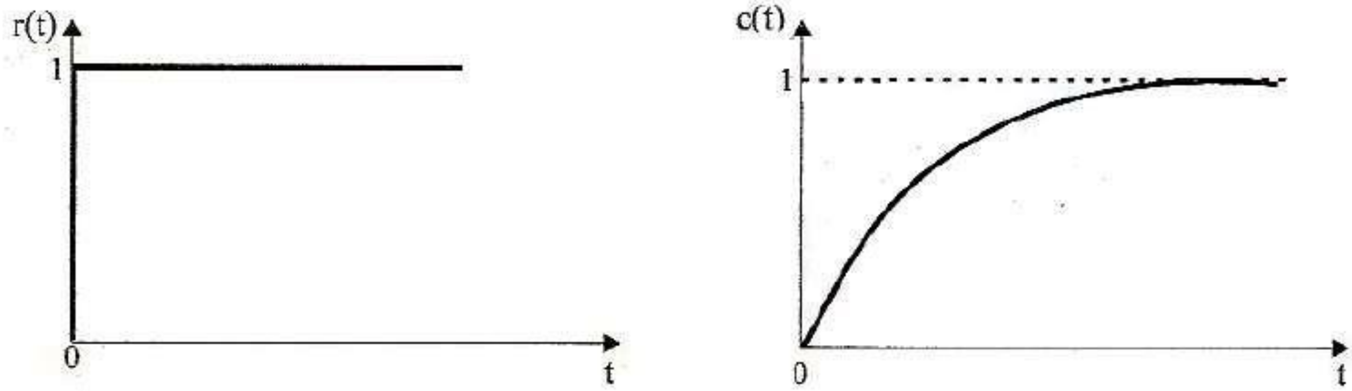

Figure 2.4.5 Response of critically damped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.25]

Case 4: Overdamped system

When $\zeta > 1$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = \{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2\}$$

$$= (s + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)}$$

For unit step input, R(s)=1/s,

$$C(s) = \frac{\omega_n^2}{s[(s + \zeta\omega_n)^2 - \omega_n^2(\zeta^2 - 1)]}$$

$$C(s) = \frac{\omega_n^2}{s(s + \zeta\omega_n + \omega_n\sqrt{1 - \zeta^2})(s + \zeta\omega_n - \omega_n\sqrt{1 - \zeta^2})}$$

By applying partial fraction,

$$C(s) = \frac{A}{s} + \frac{B}{(s + \zeta\omega_n + \omega_n\sqrt{1 - \zeta^2})} + \frac{C}{(s + \zeta\omega_n - \omega_n\sqrt{1 - \zeta^2})}$$

By applying inverse Laplace transform,

$$c(t) = [1 + \left(\frac{1}{2(\zeta + \sqrt{\zeta^2 - 1})(\sqrt{\zeta^2 - 1})}\right) e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})t}$$

$$- \left(\frac{1}{2(\zeta - \sqrt{\zeta^2 - 1})(\sqrt{\zeta^2 - 1})}\right) e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})t}]$$

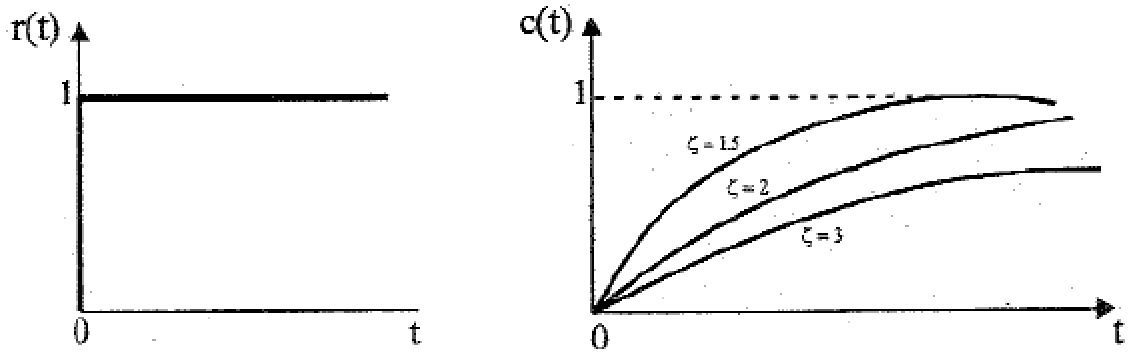

Figure 2.4.6 Response of over damped second order system to unit step input

[Source: "Control Systems" by Nagoor Kani, Page: 2.27]