Time-Domain Analysis of Control Systems

4.1 INTRODUCTION

The ability to adjust the transient and steady-state response of a control system is a beneficial outcome of the design of a feedback system. Since time is used as an independent variable in most of control systems, it is usually of interest to evaluate the state and output responses with respect to time, or simply the time response.

In the analysis problem we will use selected input signals to test the response of control system. This response will be characterized by a selected set of response measures. In this chapter, we will strive to delineate a set of quantitative performance measures that adequately represent the performance of the control systems.

4.2 TIME RESPONSE AND TEST SIGNALS

The time response of a control system is usually divided into two parts: the transient response and the steady-state response. Let $y(t)$ denote the time response of a continues-data system; then, in general, it can be written as

$$y(t) = y_t(t) + y_{ss}(t) \qquad (4.1)$$

where $y_t(t)$ denotes the transient response and $y_{ss}(t)$ denotes the steady-state response.

In control systems, transient response is defined as the part of the time response that goes to zero as time becomes very large. Thus $y_t(t)$ has the property

$$\lim_{t\to\infty} y_t(t) = 0 \qquad (4.2)$$

The steady-state response is simply the part of the total response that remains after transient has died out. All real stable systems exhibit transient phenomena to some extent before the steady state is reached.

In the design problem, specifications are usually given in terms of the transient and steady-state performance, and controllers are designed so that the specifications are all met by the design system.

Since it is difficult to design a control system so that it will perform satisfactory for all possible forms of input signals, it is necessary, for purpose of analysis and design, to assume some basic types of test signals properly for the prediction of system's performance to other more complex inputs.

1. Step-Function Input

The step-function input represents an instantaneous change in the reference input. The mathematical representation of a step function of magnitude R is

$$r(t) = \begin{cases} R & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Mathematically, $r(t) = Ru_s(t)$, where $u_s(t)$ is the unit-step function. The step function is shown in Figure 4.1 (a).

2. Ramp-Function Input

The ramp function is a signal that changes constantly with time. Mathematically, a ramp function is represented by

$$r(t) = Rtu_s(t)$$

where R is a real constant. The ramp function is shown in Figure 4.1 (b).

3. Parabolic-Function Input

The parabolic function represents a signal that is one order faster than the ramp function. Mathematically, it is represented as

$$r(t) = \frac{Rt^2}{2} u_s(t)$$

The Parabolic function is shown in Figure 4.1 (c).

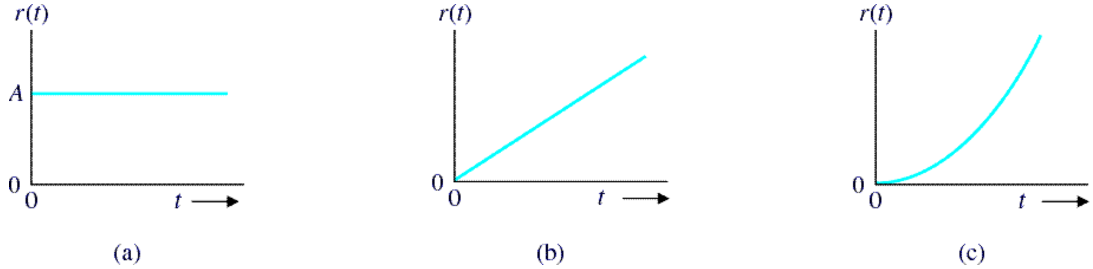

Figure 4.1 Time-domain test input signals: (a) Step, (b) Ramp, (c) Parabolic

4.3 UNIT-STEP RESPONSE AND TIME-DOMAIN SPECIFICATIONS

For linear control systems, the time response is characterized by using the unit step-input. The response of the control system to the unit step-input is called unit-step response. Figure 4.2 illustrate a typical unit-step response of a linear control system.

With reference to unit-step response, the following performance criteria (parameters) are defined:

1. Maximum overshoot

Let y_{max} denotes the maximum value of $y(t)$ and y_{ss} be the steady-state value of $y(t)$ and $y_{max} \geq y_{ss}$. The maximum overshoot of $y(t)$ is defined as,

$$\text{Maximum overshoot} = y_{max} - y_{ss}$$

$$\text{Percent maximum overshoot} = \frac{\text{maximum overshoot}}{y_{ss}} \times 100\% \qquad (4.3)$$

2. Delay time

The delay time, t_d is defined as the time required for the step response to reach 50% of its final value.

3. Rise time

The rise time, t_r is defined as the time required for the step response to rise from 10 to 90 percent of its final value.

4. Settling time

The settling time, t_s is defined as the time required for the step response to reach and stay within a specified percentage (5%) of its final value.

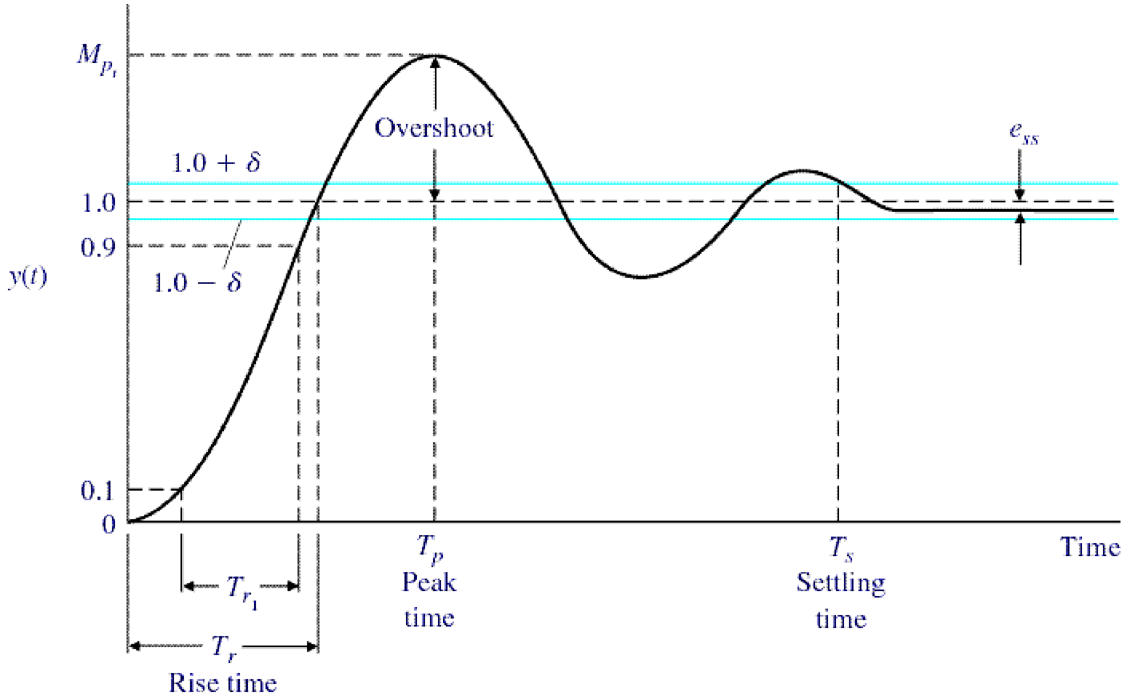

Figure 4.2 Step response of a control system

Analytically, these quantities are difficult to establish, except for simple systems lower than the third order.

4.4 TRANSIENT RESPONSE OF A PROTOTYPE SECOND-ORDER SYSTEM

Although true second-order control systems are rare in practice, their analysis generally helps to form a basis for the understanding of analysis and design of higher-order systems, especially the ones that can be approximated by second-order systems.

Consider that a second-order control system with unity feedback is represented by the block diagram shown in Figure 4.3. The open-loop transfer function of the system is

$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)} \qquad (4.5)$$

where ξ and ω_n are real constants. The closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad (4.6)$$

The characteristic equation of the prototype second-order system is obtained by setting the denominator of Eq. 4.6 to zero

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (4.7)$$

As we shall see later, the system is *stable* (*bounded output for bounded input*) if the roots of the characteristic equation locate on the left half of s-plane, and *marginally stable* (*oscillation for a bounded input*) if the characteristic equation has simple roots on the imaginary axis with all other roots in the left half of s-plane. For an *unstable* (*unbounded output for any bounded input*) system the characteristic equation has at least one root in the right half of the s-plane or it has a repeated $j\omega$ roots.

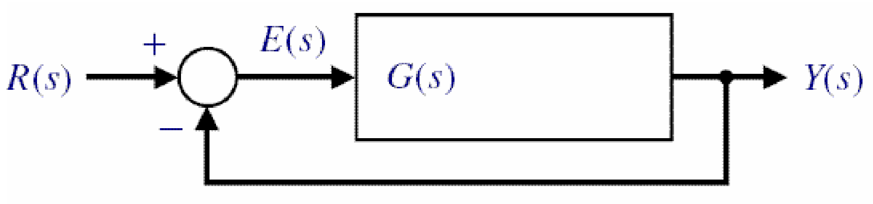

Figure 4.3 Prototype Second-order control system

For a unit-step input, R(s) = 1/s, the output response is given as

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \qquad (4.8)$$

By taking inverse Laplace transform, we obtain the unit step response of the control system

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right) \qquad t \geq 0 \qquad (4.9)$$

Figure 4.4 shows the unit-step response of the second-order system for various values of ζ. It may be noted that the response becomes more oscillatory with larger overshoot as ζ decreases.

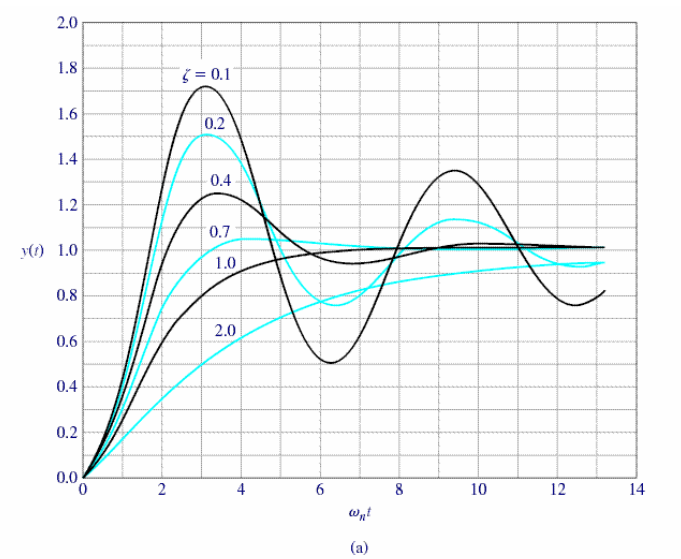

Figure 4.4 Unit-step response of second-order system with various ξ values

4.4.1 Damping Ratio and Damping Factor

The effects of the system parameters ξ and ω_n on the step response $y(t)$ can be studied by referring to the roots of the characteristic equation in Eq. (4.7). The roots can be expresses as

$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\alpha + j\omega \end{aligned} \qquad (4.10)$$

where $\alpha = \xi\omega_n$ (4.11)

and $\omega = \omega_n\sqrt{1-\zeta^2}$ (4.12)

The physical significance of ξ and α is now investigated. As seen from Eq. (4.9) the factor $\alpha = \zeta\omega_n$ appears as a constant multiplied by t in the exponential term of the response $y(t)$. Therefore, α controls the rate of rise or decay of the unit-step response $y(t)$. In other words, α controls the "damping" of the system and is called *damping factor*. The inverse of α , $1/\alpha$ is proportional to the time constant of the system. When $\zeta = 1$, the oscillations disappear and the system is said to be critically damped. Under this condition $\alpha = \omega_n$. Thus, we can regard ζ as

$$\zeta = \frac{\alpha}{\omega_n} = \frac{\textit{actual damping factor}}{\textit{damping factor at the critical damping}} \qquad (4.13)$$

When $\zeta < 1$, the system is under-damped and when $\zeta > 1$, the system is over-damped.

4.4.2 Natural Undamped Frequency

The parameter ω_n is defined as the natural undamped frequency. As seen from equation (4.10), when $\zeta = 0$, the roots of the characteristic equation are imaginary. Thus, the unit-step response of the system becomes purely oscillatory with angular frequency of ω_n. For $0 < \zeta < 1$, the imaginary parts of the roots have the magnitude of the actual (damped) frequency of oscillation.

Thus, $\omega = \omega_n\sqrt{1-\zeta^2}$

Figure 4.5 illustrates the relationships between the location of the characteristic equation roots and α, ξ, and ω_n.

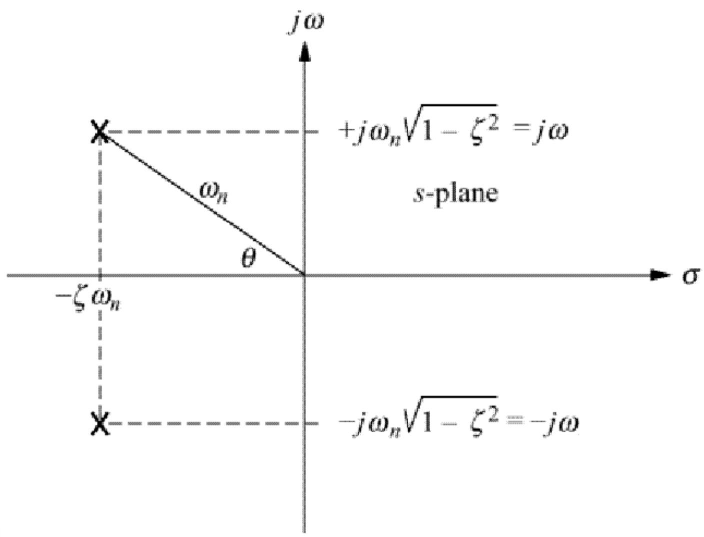

Figure 4.5 The relationship between the characteristic equation roots and α, ξ, and ω

The effect of the characteristic equation roots on the damping of the second-order system is illustrated in Figure 4.6

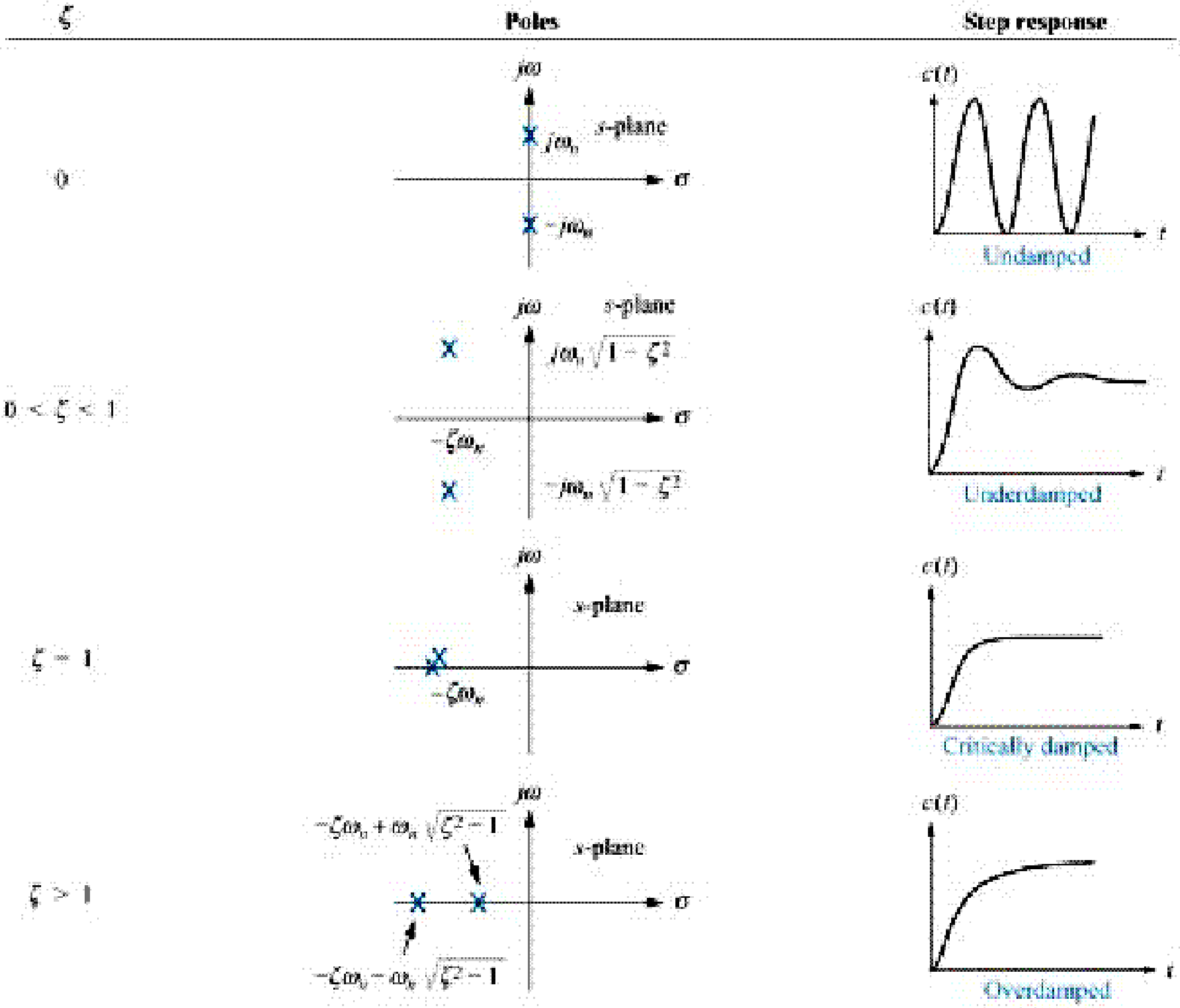

Figure 4.6 Step-response comparisons for various characteristic equation – root locations in the s-plane

4.4.3 Analytical Expression for Maximum Overshoot

By taking the derivative of Eq. (4.9) with respect to time *t* and setting the result to zero, we get

$$\frac{dy(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n . \sqrt{1-\zeta^2} . t \tag{4.14}$$

$$\omega_n \sqrt{1-\zeta^2}\, t = n\pi \qquad n = 1,2,3,...$$

From which we get

$$t = \frac{n\pi}{\omega_n \sqrt{1-\zeta^2}} \qquad n = 1,2,3,... \tag{4.15}$$

For the unit-step responses shown in Fig. 4.4, the first overshoot is the maximum overshoot. This corresponds to n = 1 in Eq. (4.15). Thus, the time at which the maximum overshoot occurs is

$$t_{\max} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \tag{4.16}$$

With reference to Fig. 4.4, the overshoots occur at odd values of n, that is, n =1, 3, 5, …, and undershoots occur at even values of n.

The magnitude of the overshoot and undershoots can be determined by subisistituting Eq. (4.14) into Eq. (4.9). This results in $y(t)_{max}$ or $y(t)_{min}$. Therefore

$$\text{maximum overshoot} = y_{max} - 1 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{4.17}$$

and the percent maximum overshoot is

$$\text{percent maximum overshoot} = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{4.18}$$

The relationship between the percent maximum overshoot and the damping ratio, given in Eq. (4.18) is plotted in Figure 4.7.

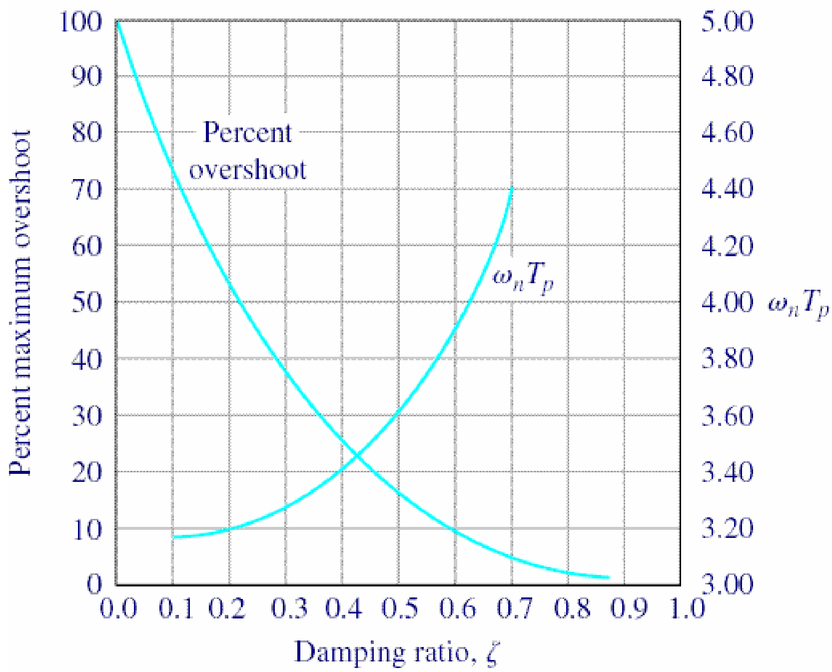

Figure 4.7 The relationship between the percent maximum overshoot and the damping ratio

4.4.4 Delay Time and Rise Time

It is more difficult to determine the exact analytical expressions of the delay time t_d and rise time t_r, and settling time t_s. However, we can utilize the linear approximation

$$t_d \cong \frac{1+0.7\zeta}{\omega_n} \qquad 0 < \zeta < 1.0 \tag{4.19}$$

The plot of $\omega_n t_r$ versus ξ is shown in Figure 4.8. This relation can be approximated by a straight line over a limited range of ξ:

$$t_r = \frac{0.60 + 2.16\zeta}{\omega_n} \qquad 0 < \zeta < 1 \tag{4.20}$$

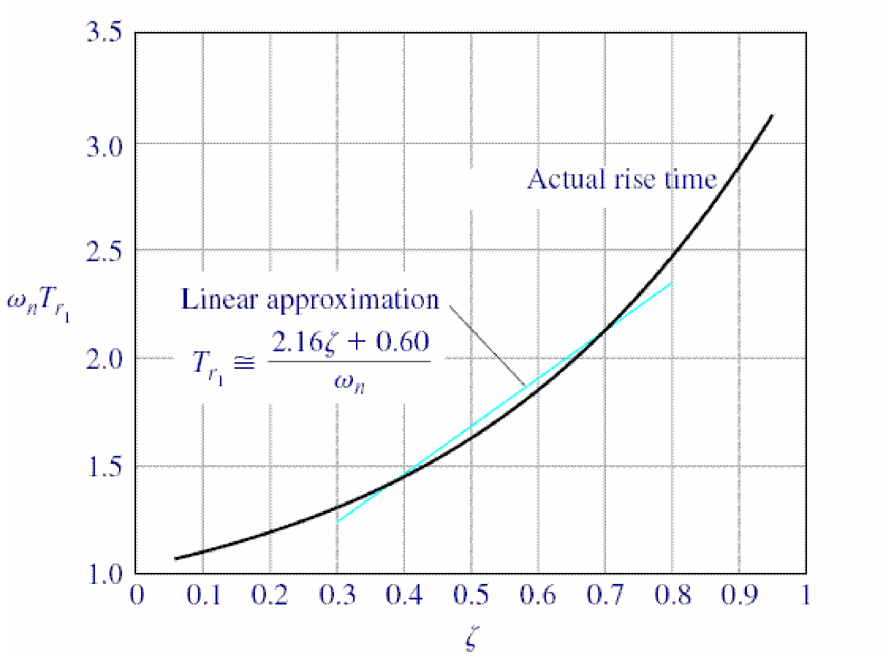

Figure 4.8 Normalized rise time versus ξ for the prototype second-order system

From this discussion, the following conclusions can be made:

1. t_r and t_d are proportional to ξ and inversely proportional to ω_n.
2. Increasing (decreasing) the natural undamped frequency ω_n will reduce (increase) t_r and t_d.

In regard to the settling time t_s, it can be approximated as

$$t_s \cong \frac{3.2}{\zeta\omega_n} \qquad 0 < \zeta < 0.69 \tag{4.21}$$

and

$$t_s = \frac{4.5\zeta}{\omega_n} \qquad \zeta > 0.69 \tag{4.22}$$

We can summarize the relationships between t_s and the system parameters as follows:

1. For $\xi < 0.69$, the settling time is inversely proportional to ξ and ω_n. A practical way of reducing the settling time is to increase ω_n while holding ξ constant.
2. For $\xi > 0.69$, the settling time is proportional to ξ and inversely proportional to ω_n. Again, t_s can be reduced by increasing ω_n.