4.1 STABILITY OF LINEAR CONTROL SYSTEMS

The transient response of a feedback control system is of primary interest and must be investigated. A stable system is defined as a system which gives a bounded output in response toa bounded input.

The concept of stability can be illustrated by considering a circular corn placed on a horizontal surface, Fig. 4.9 and Fig. 4.10.

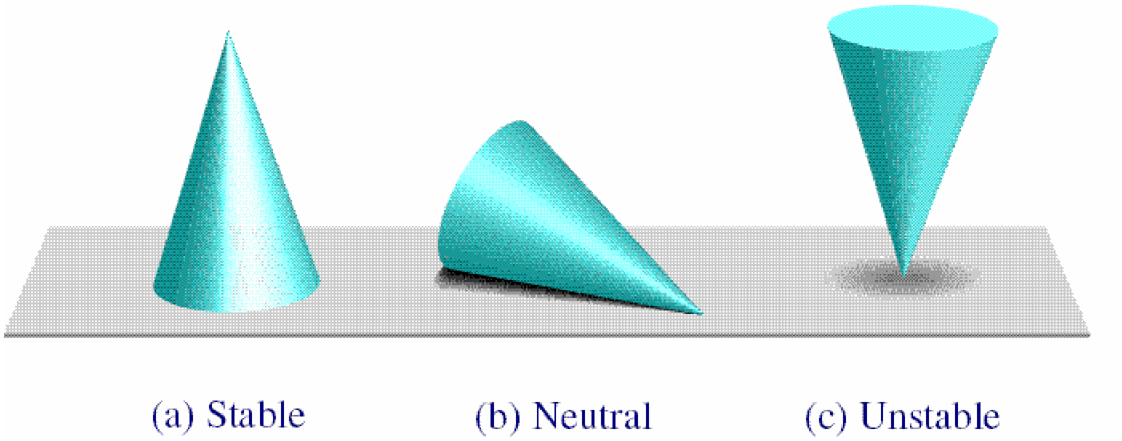

Figure 4.9 The stability of a cone

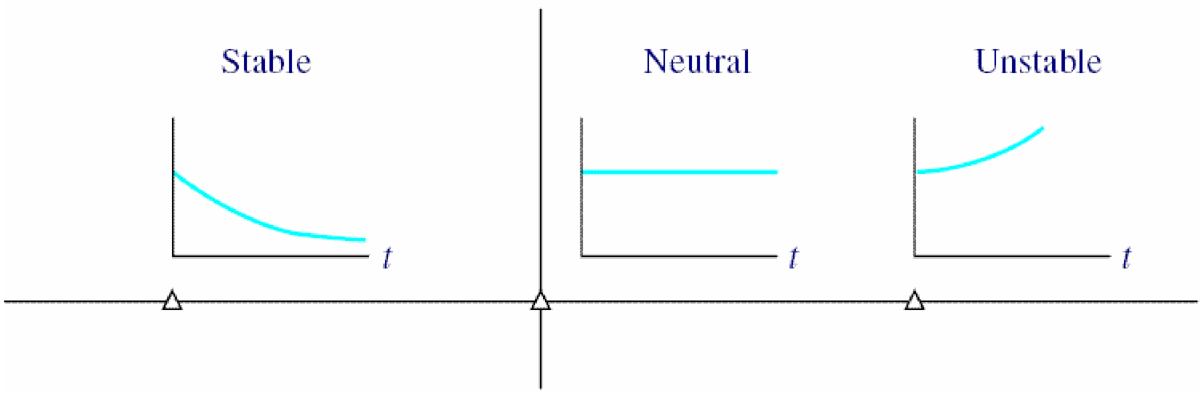

Figure 4.10 Stability in the s-plane

The stability of a dynamic system is defined in a similar manner. Let $u(t)$, $y(t)$, and $g(t)$ be the input, output, and impulse response of a linear time-invariant system, respectively. The output of the system is given by the convolution between the input and the system's impulse response.Then

$$y(t) = \int_0^{\infty} u(t-\tau)g(\tau)d\tau \qquad (4.23)$$

This response is bounded (*stable system*) if and only if the absolute value of the impulse response, $g(t)$, integrated over an infinite range, is finite. That is

$$\int_0^{\infty} |g(\tau)| d\tau \quad < \infty \qquad (4.24)$$

Mathematically, Eq. (4.24) is satisfied when the roots of the characteristic equation, or the poles of $G(s)$, are all located in the left-half s-plane.

A system is said to be *unstable* if any of the characteristic equation roots locates in the right-half *s*-plane. When the characteristic equation has simple roots on the jω-axis and none in the right- half plane, we refer to the system as *marginally stable*.

The following table illustrates the stability conditions of linear continuous system with reference to the locations of the roots of the characteristic equation.

Stability Condition	**Roots Values**
Stable	All the roots are in the left-half *s*-plane
Marginally stable or marginally unstable	At least one simple root, and no multiple roots on the jω-axis; and no roots in the right-half *s*-plane.
Unstable	At least one simple root in the right-half *s*-plane or at least one multiple-order root on the jω-axis.

Table 4.1 Stability Conditions of LTI System

The following examples illustrate the stability conditions of systems with reference to the poles of the closed-loop transfer function M(s).

$M(s) = \dfrac{20}{(s+1)(s+2)(s+3)}$	Stable
$M(s) = \dfrac{20(s+1)}{(s-1)(s^2+2s+2)}$	Unstable due to the pole at $s = 1$
$M(s) = \dfrac{20(s-1)}{(s+2)(s^2+4)}$	Marginally stable or marginally unstable due to $s = \pm j2$.
$M(s) = \dfrac{10}{(s^2+4)^2(s+10)}$	Unstable due to the multiple-order pole at $s = \pm j2$.

4.1.1 Methods of Determining Stability

The discussion in the proceeding sections lead to the conclusion that the stability of linear time-invariant system can be determined by checking on the location of the roots of the characteristic equation. When the system parameters are all known, the roots of the characteristic equation can be solved by means of a root-finding computer program. For example the M-file roots(a) of MATLAB.

For design purposes, there will be unknown or variable parameter embedded in the characteristic equation, and it will be feasible to use the root-finding programs. The method outlined below is well known for the determination of stability of LTI system without involving root solving.

4.1.1.1 Routh-Hurwitz Criterion

The Routh-Hurwitz criterion represents a method of determining the location of zeros of a polynomial with constant real coefficients with respect to the left and right half of the *s*-plane, without actually solving for the zeros.

Consider that the characteristic equation of a linear time-invariant SISO system is of the form

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{4.25}$$

where all the coefficients are real. In order that Eq. (4.25) not has roots in the right half of *s*-plane, it is *necessary* and *insufficient* that the following conditions hold:

1. All the coefficients of the equation have the same sign
2. None of the coefficients vanishes

However, these conditions are not sufficient, for it is quite possible that an equation with all its coefficients nonzero and with the same sign still may not have all the roots in the left half of the *s*-plane.

The first step in the Routh-Hurwitz criterion is to arrange the coefficients of the Eq. (4.25) as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \end{array}$$

Further rows of the schedule are then completed as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s^0 & h_{n-1} & & & \end{array}$$

where

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = -\frac{1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

and so on. Once The Routh's tabulation has been completed, we investigate the signs of the coefficients in the first column of the tabulation.

The roots of the equation are all in the left half of the *s*-plane if all the elements of the first column of the Routh's tabulation are of the same sign. The number of changes of signs in the elements of the first column equals the number of roots with positive real parts or in the right-half *s*-plane.

Example 4.1

Consider the equation

$$(s-2)(s+1)(s-3)=s^3-4s^2+s+6=0$$

This equation has one negative coefficient. Thus, we know without applying Routh's test that not all the roots of the equation are in the left-half *s*-plane. In fact, from the factored form of the equation, we know that there are two roots in the right-half *s*-plane, at $s = 2$ and $s = 3$. For purpose of illustrating the Routh's Tabulation, it is made as follows:

s^3	1	1
s^2	-4	6
s^1	2.5	0
s^0	6	0

Since there are two sign changes in the first column of the tabulation, the equation has two roots located in the right-half *s*-plane.

Example 4.2

Consider the equation

$$2s^4+s^3+3s^2+5s+10=0$$

Since this equation has no missing terms and the coefficients are all of the same sign, it satisfies the necessary conditions for not having roots in the right half or on the imaginary axis of the *s*-plane. However, since these conditions are necessary but not sufficient, we have to check Routh's tabulation.

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s^1	6.43	0	0
s^0	10	0	0

Since there are two changes in the first column of the tabulation, the equation has two roots in the right half of the *s*-plane.

4.1.1.2 Special Cases When Routh's Tabulation Terminates Prematurely

Depending on the coefficients of the equation, the following difficulties may occur that prevent Routh's tabulation from completing properly:

1. The first element in any one row of Routh's tabulation is zero, but the others are not.
2. The elements in one row of Routh's tabulation are all zero.

In the first case we replace the zero element in the first column by an arbitrary small positive number ε, and then proceed with Routh's tabulation.
This is illustrated by the following example.

Example 4.3

Consider the characteristic equation of a linear system:

$$s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

Since all the coefficients are nonzero and of the same sign, we need to apply the Routh-Hurwitz criterion. Routh's tabulation is carried out as follows:

$$\begin{array}{llll} s^4 & 1 & 2 & 3 \\ s^3 & 1 & 2 & 0 \\ s^2 & 0 & 3 & \end{array}$$

Since the first element of the s^2 row is zero, the element in the s^1 row would all be infinite. To overcome this difficulty, we replace the zero in the s^2 row by a small positive number ε and then proceed with the tabulation.

$$\begin{array}{lll} s^2 & \varepsilon & 3 \\ s^1 & \cong -\frac{3}{\varepsilon} & 0 \\ s^0 & 3 & 0 \end{array}$$

Since there are two sign changes in the first column of Routh's tabulation, the equation has two roots in the right-half *s*-plane.

In the second special case, when all the elements in one row of Routh's tabulation are zeros before the tabulation is properly terminated, it indicates that one or more of the following conditions may exist:

1. The equation has at least one pair of real roots with equal magnitude but opposite signs.
2. The equation has one or more pairs of imaginary roots.
3. The equation has pairs of complex-conjugate roots forming symmetry about the origin of the *s*-plane (e.g. $s = -1 \pm j1$, $s = 1 \pm j1$).

The situation with the entire row of zeros can be remedied by using the auxiliary equation A(s) = 0, which is formed from the coefficients of the row just above the row of zeros in Routh's tabulation. The roots of the auxiliary equation also satisfy the original equation.

To continue with Routh's tabulation when a row of zeros appears, we conduct the following steps:

1. For the auxiliary equation A(s) = 0 by use of the coefficients from the row just preceding the row of zeros.
2. Take the derivative of the auxiliary equation with respect to s; this gives $dA(s)/ds = 0$.
3. Replace the row of zeros with the coefficients of $dA(s)/ds = 0$.
4. Continue with Routh's tabulation in the usual manner.

Example 4.4

Consider the following characteristic equation of a linear control system:

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

Routh's tabulation is

$$\begin{array}{llll} s^5 & 1 & 8 & 7 \\ s^4 & 4 & 8 & 4 \\ s^3 & 6 & 6 & 0 \\ s^2 & 4 & 4 & \\ s^1 & 0 & 0 & \end{array}$$

$A(s) = 4s^2 + 4 = 0$

The derivative of $A(s)$ with respect to s is

$dA(s)/ds = 8s = 0$

From which the remaining portion of the Routh's tabulation is

$$\begin{array}{lll} s^1 & 8 & 0 \\ s^0 & 4 & \end{array}$$

Since there are no sign changes in the first column, the system is stable. Solving the auxiliary equation $A(s) = 0$, we get the two roots at $s = j$ and $s = -j$, which are also two of the roots of the characteristic equation. Thus the equation has two roots on the jω-axis, and the system ismarginally stable. These imaginary roots caused the tabulation to have an entire row of zeros in the s^1 row.

Example 4.5

Consider that a third-order control system has the characteristic equation

$$s^3 + 3408.3s^2 + 1204 \times 10^3 s + 1.5 \times 10^7 k = 0$$

Determine the crucial value of k for stability.

Routh's tabulation is

$$\begin{array}{lll} s^3 & 1 & 1204 \times 10^3 \\ s^2 & 3408 & 1.5 \times 10^7 k \\ s^1 & -\dfrac{1.5 \times 10^7 k - 3408 \times 1204 \times 10^3}{3408} & 0 \\ s^0 & 1.5 \times 10^7 & \end{array}$$

For the system to be stable, all the coefficients in the first column must have the same sign. This lead to the following conditions:

$$-\frac{1.5 \times 10^7 k - 410.36 \times 10^7}{3408} > 0$$

Therefore, the condition of k for the system to be stable is

$$0 < k < 273.57$$

If we let $k = 273.57$, the characteristic equation will have two roots on the jω-axis.
To find these roots, we substitute $k = 273.57$ in the auxiliary equation, as follows:

$$A(s) = 3408.3s^2 + 4.1036 \times 10^9 = 0$$

which has roots at s = j1097.27 and s = -j1097.27. Thus if the system operate with k = 273.57, thesystem response will be an undamped sinusoid with a frequency of 1097.27 rad/sec.