

Unit - III

Partial Differential Equations

- * Formation of partial differential Equations
- * Lagrange's Linear Equation
- * Solutions of Standard Types of first order PDE
- * Linear PDE of second order with constant coefficients

Linear PDE of 2nd and higher order with constant coefficients.

Homogeneous Linear PDEs:

A linear PDE with constant coefficients in which all the partial derivatives are of the same order is called homogeneous; otherwise it is called non-homogeneous.

Example:

Homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = \sin x$$

Non-homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial z}{\partial x} + 7 \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$$

Notation:

$$D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$$

Method of finding complementary function

Let the given equation be of the form (CF)

$$f(D, D')z = f(x, y).$$

$$\text{Put } D = m$$

$$D' = 1$$

$$f(m, 1) = 0 \Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

Let the roots of the eqn. be m_1, m_2, \dots, m_n

Roots

complementary function

1. The roots are different.

$$m_1, m_2, \dots, m_n$$

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$$

2. The roots are equal.

$$m_1 = m_2 = \dots = m_n$$

$$= m \text{ (say)}$$

$$CF = f_1(y + m x) + x f_2(y + m x) + \dots + x^{n-1} f_n(y + m x)$$

General solution is $y = CF + PI$

$$RHS = 0 \quad (Z = CF)$$

$$1. \text{ Solve } (D^2 - 6DD' + 9D'^2)Z = 0$$

Soln.:

$$\text{Put } D = m, D' = 1$$

The auxiliary equation is,

$$m^2 - 6m + 9 = 0$$

$$(m - 3)(m - 3) = 0$$

$$m = 3, 3 \text{ (equal roots)}$$

The solution is

$$Z = CF$$

$$= f_1(y + 3x) + x f_2(y + 3x)$$

$$\text{RHS} = e^{ax+by}$$

Replace D by a
 D' by b

11. Solve $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Soln:

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$\begin{matrix} (D \rightarrow m \\ D' \rightarrow 1 \end{matrix}$$

$$(m-3)(m-2) = 0$$

$$m = 2, 3 \text{ (Different)}$$

$$\begin{matrix} 6 \\ \swarrow \searrow \\ -3 \quad -2 \end{matrix}$$

$$\text{CF} = \beta_1(y+2x) + \beta_2(y+3x)$$

$$\text{PI} = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$$

$$= \frac{1}{1 - 5 + 6} e^{x+y}$$

Replace

$$D \rightarrow a = 1$$

$$D' \rightarrow b = 1$$

$$= \frac{1}{2} e^{x+y}$$

The solution is,

$$z = \text{CF} + \text{PI}$$

$$= \beta_1(y+2x) + \beta_2(y+3x) + \frac{e^{x+y}}{2}$$

12. Solve $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

Soln:

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2 \text{ (Equal)}$$

$$\begin{matrix} 4 \\ \swarrow \searrow \\ -2 \quad -2 \end{matrix}$$

$$\text{CF} = \beta_1(y+2x) + x\beta_2(y+2x)$$

$$\begin{aligned}
 \text{PI} &= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y} \\
 &= \frac{1}{2^2 - 4(2)(1) + 4(1)^2} e^{2x+y} \quad \text{Replace} \\
 & \quad D \rightarrow a = 2 \\
 & \quad D' \rightarrow b = 1 \\
 &= \frac{1}{4 - 8 + 4} e^{2x+y} \quad [\text{multiply } x \text{ in the Nr \& differentiate Dr w.r.t } D] \\
 &= x \frac{1}{2D - 4D'} e^{2x+y} \\
 &= x^2 \frac{1}{2} e^{2x+y} \quad D \rightarrow 2 \\
 & \quad D' \rightarrow 1 \\
 &= \frac{x^2}{2} e^{2x+y}
 \end{aligned}$$

The solution is $z = \text{CF} + \text{PI}$

$$\begin{aligned}
 &= f_1(y+2x) + x f_2(y+2x) \\
 & \quad + \frac{x^2}{2} e^{2x+y}
 \end{aligned}$$

Solve

1]. $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x-y}$

2]. Solve find the PI of

$$(D^2 + DD')z = e^{x-y} + e^{x+y}$$

$$\text{RHS} = \cos(ax+by) \text{ or } \sin(ax+by)$$

$$\text{Replace } D^2 \rightarrow -a^2$$

$$DD' \rightarrow -ab$$

$$D'^2 \rightarrow -b^2$$

$$\text{J. Solve } (D^2 - 2DD' + D'^2)z = \cos(x-3y)$$

Soln.:

$$\text{AE } m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, 1 \text{ (equal)}$$

$$\text{CF} = f_1(y+x) + x f_2(y+x)$$

$$\text{PI} = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y) \quad a=1, b=-3$$

$$= \frac{1}{-1 - 2(-3) - 9} \cos(x-3y)$$

$$= \frac{-1}{16} \cos(x-3y)$$

$$D^2 \rightarrow -a^2 = -1^2 = -1$$

$$DD' \rightarrow -ab = -1(-3) = 3$$

$$D'^2 \rightarrow -b^2 = -(-3)^2 = -9$$

$$\therefore z = \text{CF} + \text{PI}$$

$$= f_1(y+x) + x f_2(y+x) - \frac{1}{16} \cos(x-3y)$$

Solve

$$\text{2J. } (D^2 - 4D'^2)z = \sin(2x+y)$$

Soln.:

$$\text{AE } m^2 - 4 = 0$$

$$(m+2)(m-2) = 0$$

$$m = -2, 2 \text{ (different)}$$

$$\text{CF} = f_1(y-2x) + f_2(y+2x)$$

$$PI = \frac{1}{D^2 - 4D + 4} \sin(2x+y)$$

$$a=2$$

$$b=1$$

$$D^2 \rightarrow -a^2 = -2^2$$

$$= \frac{1}{-4 - 4(-1)} \sin(2x+y)$$

$$= -4$$

$$= x \frac{1}{2D} \sin(2x+y)$$

$$DD' \rightarrow -ab = -2(1)$$

$$= -2$$

$$= \frac{x}{2} \left(\frac{-\cos(2x+y)}{2} \right)$$

$$D'^2 \rightarrow -b^2 = -1^2$$

$$= -1$$

$$PI = -\frac{x \cos(2x+y)}{4}$$

∴ The soln. is

$$z = CF + PI$$

$$z = \delta_1 (y-2x) + \delta_2 (y+2x) - \frac{x}{4} \cos(2x+y)$$

3. Find the PI of $(D^2 - 3DD' + D'^2)z = \sin x \cos y$

Soln. :-

$$PI = \frac{1}{D^2 - 3DD' + D'^2} \sin x \cos y$$

$$\text{Gm. } (D^2 - 3DD' + D'^2)z = \frac{1}{2} [\sin(x+y) +$$

$$\sin(x-y)]$$

$$PI = \frac{1}{2} \left[\frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) \right.$$

$$\left. + \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y) \right]$$

$$= \frac{1}{2} [PI_1 + PI_2] \rightarrow (1)$$

$$\begin{aligned}
 PI_1 &= \frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) & a=1, b=1 \\
 &= \frac{1}{-1 - 3(-1) - 1} \sin(x+y) & D^2 \rightarrow -a^2 = -1 \\
 &= \frac{1}{-2+3} \sin(x+y) & DD' \rightarrow -ab = -1 \\
 & & D'^2 \rightarrow -b^2 = -1 \\
 &= \sin(x+y)
 \end{aligned}$$

$$\begin{aligned}
 PI_2 &= \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y) & a=1, b=-1 \\
 &= \frac{1}{-1 - 3(1) - 1} \sin(x-y) & D^2 \rightarrow -a^2 = -1 \\
 & & DD' \rightarrow -ab = -1(-1) = 1 \\
 & & D'^2 \rightarrow -b^2 = -(-1)^2 = -1 \\
 &= \frac{1}{-5} \sin(x-y)
 \end{aligned}$$

$$\begin{aligned}
 (1) \Rightarrow PI &= \frac{1}{2} \left[8 \sin(x+y) - \frac{1}{5} \sin(x-y) \right] \\
 &= \frac{1}{2} 8 \sin(x+y) - \frac{1}{10} \sin(x-y)
 \end{aligned}$$

Q. Find the PI of $(D^2 + 4DD' - 5D'^2)x = \sin(x-2y)$

Soln:

$$\begin{aligned}
 PI &= \frac{1}{D^2 + 4DD' - 5D'^2} \sin(x-2y) & a=1, b=-2 \\
 &= \frac{1}{1 + 4(2) - 5(-4)} \sin(x-2y) & D^2 \rightarrow -a^2 = -1 = 1 \\
 & & DD' \rightarrow -ab = -1(-2) = 2 \\
 &= \frac{1}{29} \sin(x-2y) & D'^2 \rightarrow -b^2 = -(-2)^2 = -4
 \end{aligned}$$

HW

1. Find the PI of $(D^2 + 3DD' - 4D'^2)z = \sin y$

2]. Find the PI of $\frac{\partial^2 z}{\partial x^2} - 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = \sin(3x + 2y)$

$$\text{RHS} = x^m y^n$$

1. Solve $(D^2 - 4DD' + 4D'^2)z = xy$

Soln.

$$\text{AE} = m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2 \text{ (Equal)}$$

$$\text{CF} = \{y + 2x\} + x \{y + 2x\}$$

$$\text{PI} = \frac{1}{D^2 - 4DD' + 4D'^2} xy$$

$$= \frac{1}{D^2 \left[1 - \frac{4DD'}{D^2} + \frac{4D'^2}{D^2} \right]} xy$$

$$= \frac{1}{D^2 \left[1 - \left(\frac{4D'}{D} - \frac{4D'^2}{D^2} \right) \right]} xy$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{4D'}{D} - \frac{4D'^2}{D^2} \right) \right]^{-1} xy$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{4D'}{D} - \frac{4D'^2}{D^2} \right) + \dots \right] xy$$

$$(\because (1-x)^{-1} = 1 + x + x^2 + \dots)$$

$$= \frac{1}{D^2} \left[xy + \frac{4D'}{D}(xy) - 0 \right]$$

$$= \frac{1}{D^2} \left[xy + \frac{4}{D}x \right]$$

$$= \frac{1}{D^2} xy + \frac{4}{D^3}x$$

$$= \frac{x^3 y}{6} + 4 \frac{x^4}{24}$$

$$= \frac{x^3 y}{6} + \frac{x^4}{6}$$

$$\frac{1}{D^2} xy \xrightarrow{1^{st}} \frac{1}{D} \frac{x^2}{2} y \rightarrow \frac{x^3}{6} y$$

$$\frac{1}{D^3} x \rightarrow \frac{1}{D^2} \frac{x^2}{2} \rightarrow \frac{1}{D} \frac{x^3}{6} \rightarrow \frac{x^4}{24}$$

∴ The solution is, $x = CF + PI$

$$= f_1(y+2x) + x f_2(y+2x)$$

$$+ \frac{x^3 y}{6} + \frac{x^4}{6}$$

2]. Find the PI of $(D^2 - DD' - 2D'^2)x = 2x + 3y$

Soln. :

$$PI = \frac{1}{D^2 - DD' - 2D'^2} (2x + 3y)$$

$$= \frac{1}{D^2 \left[1 - \frac{D'}{D} - \frac{2D'^2}{D^2} \right]} (2x + 3y)$$

$$= \frac{1}{D^2} \left[1 - \left(\frac{D'}{D} + \frac{2D'^2}{D^2} \right) \right]^{-1} (2x + 3y)$$

$$= \frac{1}{D^2} \left[1 + \left(\frac{D'}{D} + \frac{2D'^2}{D^2} \right) + \dots \right] (2x + 3y)$$

$$= \frac{1}{D^2} \left[2x + 3y + \frac{D'}{D} (2x + 3y) \right]$$

$$= \frac{1}{D^2} [(2x+3y) + \frac{1}{D} (3)]$$

$$= \frac{1}{D^2} [(2x+3y) + \frac{3}{D}]$$

$$= \frac{1}{D^2} (2x+3y) + \frac{3}{D^3}$$

$$\frac{1}{D^2} (2x+3y) = \frac{1}{D} \left[\frac{2x^2}{2} + 3xy \right]$$

$$= \frac{x^2}{3} + \frac{3x^2y}{2}$$

$$\frac{1}{D^3} = \frac{1}{D^2} x = \frac{1}{D} \frac{x^2}{2} = \frac{x^3}{6}$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + 3 \frac{x^3}{6}$$

$$PI = \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{x^3}{2}$$