Solution of Standard types of first order PDEs.

A Partial differential Equation in which the partial derivatives occur are of the first degree, is said to be linear; otherwise it is said to be non-linear.

Standard Types:

Type 1: $F(p, q) = 0$

Type 2: $z = px + qy + f(p, q)$ [Clairaut's form]

Type 3: $f(z, p, q) = 0$

Type 4: $f_1(x, p) = f_2(y, q)$

Type I Working Rule

1). Let $z = ax + by + c$ be the complete integral

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

2). Put $b = \phi(a)$ for general solution.

3). There is no singular integral.

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1. Solve $p + q = pq \rightarrow (1)$

Soln.:

Let $z = ax + by + c \rightarrow (2)$

Complete Integral:

Diff. partially w.r.t. 'x' and 'y'

$$\frac{\partial z}{\partial x} = a \quad \Big| \quad \frac{\partial z}{\partial y} = b$$

$$p = a \quad \Big| \quad q = b$$

Subs. the above values in (1), we get

$$a+b=ab$$
$$a=ab-b$$
$$a=b(a-1) \Rightarrow b=\frac{a}{a-1}$$

The complete integral is,

$$z=ax+\left(\frac{a}{a-1}\right)y+c \rightarrow (3)$$

Singular Integral:

diff. (3) partially w.r.t a and c and equal to zero.

$$\frac{\partial z}{\partial a}=x+\left[\frac{(a-1)(1)-a(1)}{(a-1)^2}\right]y=0$$

$$\frac{\partial z}{\partial c}=1\neq 0$$

There is no singular Integral.

General Integral:

put $c=\phi(a)$ in (3)

$$z=ax+\left(\frac{a}{a-1}\right)y+\phi(a) \rightarrow (4)$$

Diff. (4) partially w.r.t 'a'

$$\frac{\partial z}{\partial a}=x+\left[\frac{(a-1)(1)-a(1)}{(a-1)^2}\right]y+\phi'(a)=0 \rightarrow (5)$$

Eliminate 'a' b/w (4) and (5), we get the general solution.

2). Solve $\sqrt{p}+\sqrt{q}=1$

Soln:
$$\sqrt{p}+\sqrt{q}=1 \rightarrow (1)$$

Let $z=ax+by+c$

Complete Integral:

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

Subs. the above values in (1), we get

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

The complete integral is,

$$z = ax + (1-\sqrt{a})^2 y + c \rightarrow (2)$$

Singular Integral:

$$\frac{\partial z}{\partial a} = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) y = 0$$

$$\frac{\partial z}{\partial c} = 1 \neq 0$$

There is no singular integral.

General Integral:

Put $c = \phi(a)$ in (2)

$$z = ax + (1-\sqrt{a})^2 y + \phi(a) \longrightarrow (3)$$

Diff. (3) partially w.r.t 'a'

$$\frac{\partial z}{\partial a} = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) y + \phi'(a) = 0 \rightarrow (4)$$

Eliminate 'a' b/w (3) and (4),

we get the general so Integral.

TYPE – II Clairaut's Form $z = px + qy + f(p, q)$

Working Rule:

Complete Integral:

Replace $p \to a$ and $q \to b$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

General Integral:

Put $b = \phi(a)$ in Complete Integral.

I. Solve $z = px + qy + pq$

Soln.:

Gvn. $z = px + qy + pq \to (1)$

Complete Integral:

$z = ax + by + ab \to (2)$ [Replace $p \to a$, $q \to b$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

$x + b = 0$ $\quad$ $y + a = 0$

$b = -x$ $\quad$ $a = -y$

Subs. a & b in (2)

$z = -yx - xy - y(-x)$

$z = -xy$

General Integral:

Subs. $b = \phi(a)$ in (2),

$z = ax + \phi(a)y + a\phi(a) \to (3)$

WKT $\frac{\partial z}{\partial a} = 0$

$\Rightarrow x + \phi'(a)y + a\phi'(a) + \phi(a) = 0 \to (4)$

Eliminate 'a' b/w (4) & (3), we get the general Soln.

Q. Solve $z = px + qy + p^2 - q^2$

Soln.:

Givn. $z = px + qy + p^2 - q^2 \rightarrow (1)$

Complete Integral:

$z = ax + by + a^2 - b^2$ Replace $p \rightarrow a$, $q \rightarrow b$

$\hookrightarrow (2)$

Singular Integral:

$\frac{\partial z}{\partial a} = 0$

$x + 2a = 0$

$2a = -x$

$a = \frac{-x}{2}$

$\frac{\partial z}{\partial b} = 0$

$y - 2b = 0$

$y = 2b$

$\Rightarrow b = \frac{y}{2}$

Subs. a & b in (2),

$z = -\frac{x}{2}x + \frac{y}{2}y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$

$= -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$

$4z = -2x^2 + 2y^2 + x^2 - y^2$

$= -x^2 + y^2$

$y^2 - x^2 = 4z$

General Integral:

Subs. $b = \phi(a)$ in (2)

$\rightarrow (3)$

$z = ax + \phi(a)y - [\phi(a)]^2 + a^2$

WKT $\frac{\partial z}{\partial a} = 0$

$\Rightarrow x + \phi'(a)y - 2\phi(a)\phi'(a) + 2a = 0 \rightarrow (4)$

Eliminate 'a' b/w (3) & (4), we get the general soln.

Type - III $\quad f(z, p, q) = 0$

1. Solve $\quad p(1+q) = qz \rightarrow (1)$

Soln. :

Let $u = x + ay$

Then $\quad p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

$$(1) \Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

$$\frac{du}{dz} = \frac{a}{az-1}$$

$$du = \frac{a}{az-1}\,dz$$

Integrating, $\quad u = \int \frac{a}{az-1}\,dz$

$$u = \log(az-1) + \log c$$

$$x + ay = \log[c(az-1)]$$

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2. Solve $\quad z^2 = 1 + p^2 + q^2$

Soln.

$$z^2 = 1 + p^2 + q^2 \rightarrow (1)$$

Let $u = x + ay$

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

$$(1) \Rightarrow \quad z^2 = 1 + \left(\frac{dz}{du}\right)^2 + \left(a\,\frac{dz}{du}\right)^2$$

$$z^2 = \left(\frac{dz}{du}\right)^2 (1+a^2) + 1$$

$$z^2 - 1 = \left(\frac{dz}{du}\right)^2 (1+a^2)$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{1+a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{z^2-1}{1+a^2}} = \frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$$

$$\frac{dz}{\sqrt{z^2-1}} = \frac{du}{\sqrt{1+a^2}}$$

Integrating on both sides,

$$\cosh^{-1} z = \frac{1}{\sqrt{1+a^2}}\,u + c$$

$$= \frac{1}{\sqrt{1+a^2}}(x+ay) + c$$

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Type - IV $\quad f_1(x, p) = f_2(y, q)$

For this type, there is no Singular Integral.

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1. Solve $q^2 - p = y - x$

Soln. :

Given. $q^2 - y = p - x = k$ (a constant)

Now $q^2 - y = k$ | $p - x = k$

$q^2 = k + y$ | $p = k + x$

$q = \sqrt{k+y}$

we know that $z = \int p\,dx + \int q\,dy$

$$z = \int (k+x)\,dx + \int \sqrt{k+y}\,dy$$

$$= kx + \frac{x^2}{2} + \frac{(k+y)^{3/2}}{3/2} + c$$

$$= kx + \frac{x^2}{2} + \frac{2}{3}(k+y)^{3/2} + c,$$ which is the complete Integral.

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2. Solve $\sqrt{p} + \sqrt{q} = x + y$

Soln.

Given. $\sqrt{p} - x = y - \sqrt{q} = k$

Now $\sqrt{p} - x = k$ | $y - \sqrt{q} = k$

$\sqrt{p} = k + x$ | $\sqrt{q} = y - k$

$p = (k+x)^2$ | $q = (y-k)^2$

we know that

$$z = \int p\,dx + \int q\,dy$$

$$z = \int (k+x)^2 dx + \int (y-k)^2 dy$$

$$= \frac{(k+x)^3}{3} + \frac{(y-k)^3}{3} + c,$$ which is the complete Integral

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3]. Find the complete Integral of

$xp - yq = y^2 - x^2$

Soln.

Gvn. $xp + x^2 = y^2 + yq = k$ (a constant)

Now $xp + x^2 = k$

$xp = k - x^2$

$p = \frac{k - x^2}{x}$

$p = \frac{k}{x} - x$

$y^2 + yq = k$

$yq = k - y^2$

$q = \frac{k - y^2}{y}$

$q = \frac{k}{y} - y$

we know that $z = \int p\, dx + \int q\, dy$

$$z = \int \left(\frac{k}{x} - x\right) dx + \int \left(\frac{k}{y} - y\right) dy$$

$$= k \log x - \frac{x^2}{2} + k \log y - \frac{y^2}{2} + c$$

$$= k \log xy - \left(\frac{x^2 + y^2}{2}\right) + c$$ which is the CI.

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1. Form the PDE from $z = ax + by + \sqrt{a^2+b^2}$

Soln.:

Given $z = ax + by + \sqrt{a^2+b^2} \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a + 0 + 0$$

$$\Rightarrow \quad p = a \rightarrow (2)$$

Differentiate partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 0 + b + 0$$

$$\Rightarrow \quad q = b \longrightarrow (3)$$

Subs. (2) and (3) in (1),

$$z = px + qy + \sqrt{p^2+q^2}$$

2. Form the PDE from $ax^2 + by^2 + z^2 = 1$

Soln.:

Given. $ax^2 + by^2 + z^2 = 1 \rightarrow (1)$

Differentiate partially w.r.t 'x'

$$2ax + 0 + 2z\frac{\partial z}{\partial x} = 0$$

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$$2ax = -2zp$$

$$a = \frac{-zp}{x}$$

Differentiate partially w.r.t 'y'

$$0 + 2by + 2z\frac{\partial z}{\partial y} = 0$$

$$2by + 2zq = 0$$

$$2by = -2zq$$

$$b = \frac{-zq}{y}$$

Subs. a and b in (1),

$$-\frac{zp}{x}x^2 - \frac{zq}{y}y^2 + z^2 = 1$$

$$-zpx - zqy + z^2 = 1$$

$$z(z - px - qy) = 1$$

iv). $z = f(x+t) + g(x-t)$

$$p = f'(x+t) + g'(x-t)$$

$$q = f'(x+t) - g'(x-t)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+t) + g''(x-t) \longrightarrow (1)$$

$$s = \frac{\partial^2 z}{\partial x \partial t} = f''(x+t) - g''(x-t) \longrightarrow (2)$$

$$t = \frac{\partial^2 z}{\partial t^2} = f''(x+t) + g''(x-t) \longrightarrow (3)$$

From (1) and (3), $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$