

Lagrange's Linear Equation:

The equation is of the form

$Pp + Qq = R$, where P , Q and R are functions of x, y, z . This is known as Lagrange's linear eqn.

To solve this equation, it is enough to solve the subsidiary (or) auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The auxiliary eqn. can be solved in two ways.

- i). method of grouping
- ii). method of multipliers.

method of grouping:

In the auxiliary eqn., $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

If the variables can be separated in any pair of eqns., then we get a solution of the form

$$u(x, y) = C_1 \quad \text{and} \quad v(x, y) = C_2$$

i.e., $\phi(u, v) = 0$ where ϕ is arbitrary.

II. Solve $Px^2 + Qy^2 = z^2$

Soln.:

$$Px^2 + Qy^2 = z^2$$

This eqn. is of the form

$$Pp + Qq = R \quad \text{where} \quad P = x^2, \quad Q = y^2$$

$$\text{and} \quad R = z^2$$

The auxiliary eqn. is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Take $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating, we get

$$\int x^{-2} dx = \int y^{-2} dy$$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + C_1$$

$$-\frac{1}{x} + \frac{1}{y} = C_1$$

$$u = \frac{1}{y} - \frac{1}{x}$$

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

$$\int y^{-2} dy = \int z^{-2} dz$$

$$\frac{y^{-1}}{-1} = \frac{z^{-1}}{-1} + C_2$$

$$\frac{1}{y} + \frac{1}{z} = C_2$$

$$v = \frac{1}{z} - \frac{1}{y}$$

∴ The soln. is $\phi(u, v) = 0$

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$$

7. Solve $\frac{y^2 z}{x} p + x z q = y^2$

Soln.:

$$\frac{y^2 z}{x} p + x z q = y^2$$

$$P = \frac{y^2 z}{x}, \quad Q = x z$$

$$R = y^2$$

AE

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{x z} = \frac{dz}{y^2}$$

$$\frac{dx}{y^2 z} = \frac{dy}{xz}$$

$$\frac{x dx}{y^2 z} = \frac{dy}{xz}$$

$$x^2 dx = y^2 dy$$

Integrating,

$$\frac{x^3}{3} = \frac{y^3}{3} + C_3$$

$$\frac{x^3}{3} - \frac{y^3}{3} = C_3$$

$$x^3 - y^3 = 3C_3 = u$$

\therefore The soln. is $\phi(u, v) = 0$

$$\phi(x^3 - y^3, x^2 - z^2) = 0$$

2]. Solve $Px + Qy = R$

Soln.

$$Px + Qy = R \Rightarrow P = x, Q = y, R = x$$

$$\frac{AE}{x} \quad \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log C_1$$

$$\log x - \log y = \log C_1$$

$$\log\left(\frac{x}{y}\right) = \log C_1$$

$$\frac{x}{y} = C_1 \Rightarrow u = \frac{x}{y}$$

$$\Rightarrow \phi\left(\frac{x}{y}, x - z\right) = 0$$

$$\frac{dx}{y^2 z} = \frac{dz}{y^2}$$

$$\frac{x dx}{y^2 z} = \frac{dz}{y^2}$$

$$x dx = dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + C_2$$

$$x^2 - z^2 = 2C_2$$

$$v = x^2 - z^2$$

4]. Solve $P \tan x + Q \tan y = \tan z$

Soln.:

$$Pp + Qq = R \Rightarrow P = \tan x, Q = \tan y, R = \tan z \quad \neq$$

AE

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating,

$$\int \cot x dx = \int \cot y dy$$

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$\log(\sin x) - \log(\sin y) = \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

$$\Rightarrow u = \frac{\sin x}{\sin y}$$

$$\Rightarrow \phi(u, v) = 0$$

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \cot y dy = \int \cot z dz$$

$$\log(\sin y) = \log(\sin z) + \log c_2$$

$$\log(\sin y) - \log(\sin z) = \log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$$\Rightarrow v = \frac{\sin y}{\sin z}$$

5]. Solve $P\sqrt{x} + Q\sqrt{y} = \sqrt{z}$

Soln.:

$$Pp + Qq = R \Rightarrow P = \sqrt{x}, Q = \sqrt{y}, R = \sqrt{z}$$

$$\frac{AE}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$2\sqrt{x} = 2\sqrt{y} + 2C_1$$

$$\sqrt{x} = \sqrt{y} + C_1$$

$$\sqrt{x} - \sqrt{y} = C_1$$

$$\Rightarrow u = \sqrt{x} - \sqrt{y}$$

$$\Rightarrow \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$$

$$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$2\sqrt{y} = 2\sqrt{z} + 2C_2$$

$$\sqrt{y} = \sqrt{z} + C_2$$

$$\sqrt{y} - \sqrt{z} = C_2$$

$$\Rightarrow v = \sqrt{y} - \sqrt{z}$$

Hw

1]. $Px + Qy = \sqrt{z}$

2]. $x^2 P + y^2 Q = z^2$

3]. $P - Q = \log(x + y)$