

Method of multipliers:

Choose any 3 multipliers l, m, n which may be constant (or) functions of x, y, z . Then

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$, then $l dx + m dy + n dz = 0$

Direct integration gives $u(x, y, z) = C_1$

Similarly choose another set of 3 multipliers l', m' and n' ,

$$v(x, y, z) = C_2$$

$$\Rightarrow \phi(u, v) = 0$$

$$\sqrt{1} \text{ J. Solve } x(x^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$$

Soln. :

$$x(x^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2) \rightarrow (1)$$

This eqn. is of the form,

$$Pp + Qq = R$$

$$\text{where } P = x(x^2 - y^2); \quad Q = y(x^2 - z^2); \quad R = z(y^2 - x^2)$$

$$\frac{AE}{x(x^2 - y^2)} \frac{dx}{x(x^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)}$$

Choosing x, y, z as Lagrange's multipliers,

$$\frac{x dx + y dy + z dz}{x^2(x^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)}$$

$$= \frac{x dx + y dy + z dz}{x^2 x^2 - x^2 y^2 + y^2 x^2 - y^2 z^2 + y^2 x^2 - x^2 z^2}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$\text{i.e., } x dx + y dy + z dz = 0$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

$$x^2 + y^2 + z^2 = 2C_1$$

$$\Rightarrow u = x^2 + y^2 + z^2$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as Lagrange's multipliers

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$$

$$\frac{1}{x} x(x^2 - y^2) + \frac{1}{y} y(x^2 - z^2) + \frac{1}{z} z(y^2 - x^2)$$

$$= \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{x^2 - y^2 + x^2 - z^2 + y^2 - x^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\text{ie, } \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log C_2$$

$$\log (xyz) = \log C_2$$

$$\Rightarrow C_2 = xyz$$

$$\Rightarrow V = xyz$$

The solution is, $\phi(u, v) = 0$

$$\phi(x^2 + y^2 + z^2, xyz) = 0$$

Q7. Solve $(mx - ny)p + (nx - lz)q = ly - mx$

Soln.:

$$(mx - ny)p + (nx - lz)q = ly - mx \rightarrow (1)$$

This eqn. is of the form

$$Pp + Qq = R$$

where $P = (mx - ny)$; $Q = (nx - lz)$; $R = ly - mx$

$$\frac{AE}{mx - ny} \frac{dx}{dx} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Choosing l, m, n as Lagrange's multipliers,

$$l dx + m dy + n dz$$

$$l(mx - ny) + m(nx - lz) + n(ly - mx)$$

$$= \frac{ldx + mdy + ndz}{lmz - lny + mnx - lmz + lny - mnx}$$

$$= \frac{ldx + mdy + ndz}{0}$$

ie, $ldx + mdy + ndz = 0$

Integrating, we get

$$lx + my + nz = C_1$$

$$\Rightarrow u = lx + my + nz$$

choosing x, y, z as Lagrange's multipliers,
we get

$$\frac{x dx + y dy + z dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{x dx + y dy + z dz}{mxz - nxy + nxy - lyz + lyz - mxz}$$

$$= \frac{x dx + y dy + z dz}{0}$$

ie, $x dx + y dy + z dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

$$x^2 + y^2 + z^2 = 2C_2$$

$$\Rightarrow v = x^2 + y^2 + z^2$$

The solution is, $\phi(u, v) = 0$

$$\phi(lx + my + nz, x^2 + y^2 + z^2) = 0$$

3]. Solve $(3x - 4y)P + (4x - 2z)Q = 2y - 3x$

Soln.:

This eqn. is of the form

$$Pp + Qq = R$$

Here $P = 3x - 4y$; $Q = 4x - 2z$; $R = 2y - 3x$

AE

$$\frac{dx}{3x - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$$

choosing 2, 3, 4 as lagrange's multipliers,

$$\begin{aligned} & \frac{2dx + 3dy + 4dz}{2(3x - 4y) + 3(4x - 2z) + 4(2y - 3x)} \\ &= \frac{2dx + 3dy + 4dz}{6x - 8y + 12x - 6z + 8y - 12x} \\ &= \frac{2dx + 3dy + 4dz}{0} \end{aligned}$$

i.e., $2dx + 3dy + 4dz = 0$

Integrating, we get

$$2x + 3y + 4z = C_1$$

$$\Rightarrow u = 2x + 3y + 4z$$

choosing x, y, z as lagrange's multipliers,

$$\begin{aligned} & \frac{x dx + y dy + z dz}{x(3z - 4y) + y(4x - 2z) + z(2y - 3x)} \\ &= \frac{x dx + y dy + z dz}{3xz - 4xy + 4xy - 2yz + 2yz - 3xz} \\ &= \frac{x dx + y dy + z dz}{0} \end{aligned}$$

i.e., $x dx + y dy + z dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{1}{2} C_2$$

$$x^2 + y^2 + z^2 = C_2$$

$$\Rightarrow v = x^2 + y^2 + z^2$$

The solution is, $\phi(u, v) = 0$

$$\phi(2x + 3y + 4z, x^2 + y^2 + z^2) = 0$$

Q. Solve $x(y^2 + z)P - y(x^2 + z)Q = z(x^2 - y^2)R$

Soln.:

$$Pp + Qq = Rr$$

Here $P = x(y^2 + z)$; $Q = -y(x^2 + z)$;
 $R = z(x^2 - y^2)$

$$\frac{AE}{dx} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)} \rightarrow (2)$$

choosing x, y and -1 as Lagrange's multipliers, each ratio of (2) is equal to

$$\begin{aligned} & \frac{x dx + y dy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} \\ &= \frac{x dx + y dy - dz}{x^2 y^2 + x^2 z - x^2 y^2 - y^2 z - x^2 z + y^2 z} \\ &= \frac{x dx + y dy - dz}{0} \end{aligned}$$

ie, $x dx + y dy - dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = C_1$$

$$x^2 + y^2 - 2z = 2C_1$$

$$\text{i.e., } u = x^2 + y^2 - 2z$$

choosing $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ as Lagrange's multipliers,

$$\frac{dx}{x} + \frac{dy}{y} \neq \frac{dz}{z}$$

$$\frac{1}{x} x (y^2 + z) - \frac{1}{y} y (x^2 + z) + \frac{1}{z} z (x^2 - y^2)$$

$$= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

$$y^2 + z - x^2 - z + x^2 - y^2$$

$$= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

0

$$\text{i.e., } \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log C_2$$

$$\log xyz = \log C_2$$

$$\Rightarrow C_2 = xyz$$

$$\Rightarrow v = xyz$$

The solution is,

$$\phi(u, v) = 0$$

$$\phi(x^2 + y^2 - 2z, xyz) = 0$$

Hw

1]. Solve $x(y-z)p + y(z-x)q = z(x-y)$

2]. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

3]. Solve $(y^2+z^2)p - xyq + xz = 0$

4]. Solve $(x^2-yz)p + (y^2-zx)q = z^2 - xy$
