Method of Multipliers

$Pp+Qq=R$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

I $$\frac{dx\pm dy\pm dz}{P\pm Q\pm R}=0$$

II $$\frac{x\,dx\pm y\,dy\pm z\,dz}{xP\pm yQ\pm zR}=0$$

III $$\frac{x^2dx\pm y^2dy\pm z^2dz}{x^2P\pm y^2Q\pm z^2R}=0$$

IV $$\frac{\frac{1}{x}dx\pm\frac{1}{y}dy\pm\frac{1}{z}dz}{\frac{1}{x}P\pm\frac{1}{y}Q\pm\frac{1}{z}R}=0$$

V $$\frac{\frac{1}{x^2}dx\pm\frac{1}{y^2}dy\pm\frac{1}{z^2}dz}{\frac{1}{x^2}P\pm\frac{1}{y^2}Q\pm\frac{1}{z^2}R}=0$$

VI $$\frac{l\,dx\pm m\,dy\pm n\,dz}{lP\pm mQ\pm nR}=0$$

1. Solve $x(y-z)p+y(z-x)q=z(x-y)$

$Pp+Qq=R$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$$

I $\Rightarrow$ $\dfrac{dx+dy+dz}{xy-xz+yz-xy+zx-zy}=0$

$dx+dy+dz=0$

$\int dx+\int dy+\int dz=0$

$x+y+z=c_1 \rightarrow (1)$

IV $\Rightarrow$ $\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}\cdot x(y-z)+\frac{1}{y}y(z-x)+\frac{1}{z}z(x-y)}=0$

$\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y}=0$

$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0.$

$\int\frac{1}{x}dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=0$

$\log x+\log y+\log z=\log c_2$

$\log(xyz)=\log c_2$

$xyz=c_2$

$\phi(c_1,c_2)=0$

$\phi(x+y+z,xyz)=0.$

2. Solve $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$

$Pp+Qq=R$

$\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$

$\dfrac{dx}{x(z^2-y^2)}=\dfrac{dy}{y(x^2-z^2)}=\dfrac{dz}{z(y^2-x^2)}$

$$\text{II} \Rightarrow \frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2) + y^2(x^2-z^2) + z^2(y^2-x^2)} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + y^2z^2 - x^2z^2} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1 \rightarrow (1)$$

$$\text{IV} \Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(z^2-y^2) + \frac{1}{y}\cdot y(x^2-z^2) + \frac{1}{z}\cdot z(y^2-x^2)} = 0$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{z^2-y^2+x^2-z^2+y^2-x^2} = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0.$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2 \rightarrow (2)$$

By combining (1) & (2)

$$\phi(c_1, c_2) = 0$$

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, xyz\right) = 0.$$

3. Solve $(mz-ny)p+(nx-lz)q=ly-mx$

$$Pp+Qq=R$$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

$$\text{II} \Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=0$$

$$\frac{x\,dx+y\,dy+z\,dz}{xmz-xny+ynx-lyz+lyz-mxz}=0$$

$$x\,dx+y\,dy+z\,dz=0$$

$$\int x\,dx+\int y\,dy+\int z\,dz=0.$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_1 \rightarrow (1)$$

$$\text{IVI} \Rightarrow \frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=0$$

$$\frac{l\,dx+m\,dy+n\,dz}{lmz-lny+mnx-lmz+lny-mnx}=0$$

$$l\,dx+m\,dy+n\,dz=0$$

$$\int l\,dx+\int m\,dy+\int n\,dz=0$$

$lx+my+nz = c_2 \rightarrow (2)$.

Combining (1) & (2),

$$\phi(c_1, c_2) = 0$$

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, lx+my+nz\right) = 0.$$

(4) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$Pp+Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{x^2}\cdot x^2(y-z) + \frac{1}{y^2}\cdot y^2(z-x) + \frac{1}{z^2}\cdot z^2(x-y)} = 0.$$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y} = 0.$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$$\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = 0.$$

$$\left(\frac{-1}{x}\right) + \left(\frac{-1}{y}\right) + \left(\frac{-1}{z}\right) = c_1$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}\cdot x^2(y-z) + \frac{1}{y}\cdot y^2(z-x) + \frac{1}{z}\cdot z^2(x-y)} = 0$$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)}=0.$$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{xy-xz+yz-xy+xz-yz}=0.$$

$$\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0.$$

$$\int\frac{1}{x}dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=0.$$

$$\log x+\log y+\log z=\log c_2$$

$$\log(xyz)=\log c_2$$

$$xyz=c_2. \rightarrow (2)$$

By combining ① & ②

$$\phi(c_1,c_2)=0$$

$$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},xyz\right)=0.$$

⑤ Solve: $z(x-y)=x^2p-y^2q$

$Pp+Qq=R$ $\quad P=x^2 \quad Q=-y^2 \quad R=z(x-y)$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x^2}=\frac{dy}{-y^2}=\frac{dz}{z(x-y)}=\frac{dx+dy}{(x+y)(x-y)}$$

Taking first two members

$$\frac{dx}{x^2}=\frac{dy}{-y^2}$$

Integrating $\int \frac{1}{x^2} dx = -\int \frac{1}{y^2} dy$

$$\frac{-1}{x} = \frac{1}{y} - c_1$$

$$c_1 = \frac{1}{x} + \frac{1}{y}$$

Taking last two members

$$\frac{dz}{z(x-y)} = \frac{dx+dy}{(x+y)(x-y)}$$

$$\frac{dz}{z} = \frac{d(x+y)}{(x+y)}$$

Integrating, $\int \frac{dz}{z} = \int \frac{d(x+y)}{(x+y)}$

$$\log z = \log(x+y) + \log c_2.$$

$$\log z - \log(x+y) = \log c_2$$

$$\log\left(\frac{z}{x+y}\right) = \log c_2$$

$$c_2 = \frac{z}{x+y}$$

$\therefore \phi(c_1, c_2) = 0$

$$\phi\left(\frac{1}{x} + \frac{1}{y}, \frac{z}{x+y}\right) = 0.$$

⑥ Solve $(y^2+z^2)p - xyq + xz = 0.$

$Pp + Qq = R$ $\quad P = y^2+z^2,\ Q = -xy \quad z = -xz$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

I $\quad \frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) - y(xy) - z(xz)} = 0$

$$\frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - xy^2 - xz^2} = 0$$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0.$

$\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$x^2 + y^2 + z^2 = c_1$

$\Rightarrow \quad \frac{dy}{-xy} = \frac{dz}{-xz}$

$\frac{dy}{y} = \frac{dz}{z}$

Integrating, $\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_2$

$\log y - \log z = \log c_2$

$\log\left(\frac{y}{z}\right) = \log c_2$

$c_2 = \frac{y}{z}$

$\Rightarrow \phi(c_1, c_2) = 0$

$\phi\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0.$

7) Solve $(x+2z)p + (2xz - y)q = x^2 + y.$

$P = x + 2z \quad Q = 2xz - y \quad R = x^2 + y$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x+2z} = \frac{dy}{2xz-y} = \frac{dz}{x^2+y}$ [Multipliers : $y, x, -2z$ & $2x, -2y, -2z$]

$$\text{I} \Rightarrow \frac{ydx + xdy - 2zdz}{y(x+2z) + x(2xz-y) - 2z(x^2+y)}$$

$$\frac{ydx + xdy - 2zdz}{xy + 2yz + 2x^2z - xy - 2x^2z - 2yz}$$

$$ydx + xdy - 2zdz = 0.$$

$$d(xy - z^2) = 0$$

Integrating $\int d(xy - z^2) = 0.$

$$xy - z^2 = c_1$$

$$\text{II} \Rightarrow \frac{2xdx - 2dy - 2dz}{2x(x+2z) - 2(2xz+y) - 2(x^2+y)}$$

$$\frac{2xdx - 2dy - 2dz}{2x^2 + 4xz - 4xz - 2y - 2x^2 - 2y}$$

$$2xdx - 2dy - 2dz = 0.$$

$$d\left(\frac{2x^2}{2} - 2y - 2z\right) = 0$$

$$\int d(x^2 - 2y - 2z) = 0$$

$$x^2 - 2y - 2z = c_2$$

$$\phi(c_1, c_2) = 0$$

$$\phi(xy - z^2,\ x^2 - 2y - 2z) = 0.$$