DEPARTMENT OF MATHEMATICS

PUZZLE-3

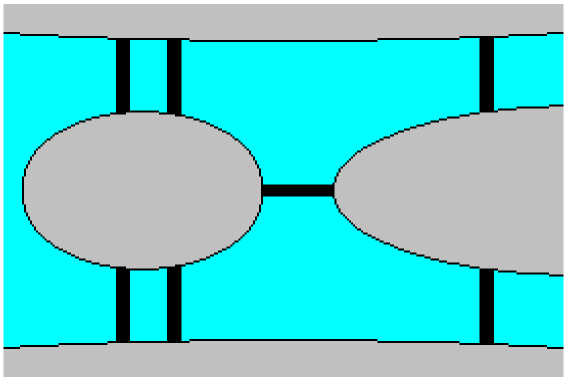
Hamilton Paths and Cycles on puzzles

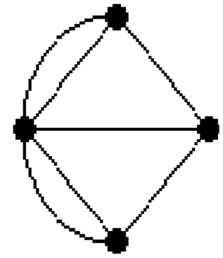
I have always wondered whether it is possible to find a sequence of moves on the Rubik's Cube that visits every position exactly once. My attempts to answer this question form the basis of this page. Though this is of little or no practical value in the real world of puzzles, I hope you find it interesting. Some basic knowledge of group theory is assumed, but you can probably understand the gist of most things from the context.

Koenigsberg's bridges and Eulerian paths

Given the arrangement of 7 bridges between 4 landmasses shown on the right. Is it possible to devise a route that crosses each bridge exactly once?

This was the arrangement of bridges in the city of Koenigsberg at the time Leonhard Euler solved the problem. He realised that if a landmass has an odd number of bridges leading from it, then the journey must start or end there. Every time you enter or leave a landmass, you use up one of its bridges. With an odd number of bridges, you must enter that landmass more often than leaving it, or vice versa. That can only mean you start there and end somewhere else, or end there but start somewhere else. In the case of Koenigsberg all four landmasses have an odd number of bridges, so no single route can cross every bridge.





The diagram of the city can be simplified to a graph of four vertices (the landmasses) and 7 edges (bridges) connecting them. An **Eulerian path** on a graph is a path that includes every edge exactly once. Euler proved that a graph must have no more than two vertices with an odd number of edges for there to be an Eulerian path. An **Eulerian cycle** or Eulerian circuit is an path that uses up every edge exactly once and also ends at the same vertex as it started. For an Eulerian cycle to exist, all the vertices of the graph must have an even number of edges.

The converse of these facts is also true, provided the graph is connected (i.e. does not consist of several separate subgraphs). If there are two odd vertices in a connected graph, then there is a path that goes along all the edges that starts and ends at those two vertices. If there are no odd vertices, then there is a path that goes along all the edges, and that path will have to start and end at the same vertex. These Eulerian paths or cycles are easy to find, because you can take any route you like provided that you make sure that you never do a move that would cause the untravelled edges to fall apart into two or more disconnected parts.