UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

Type 2: problems based on vibrating string with non-zero initial velocity:

1. A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$. Find the displacement.

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

i) $y(0,t)=0, \ \forall t$

(ii) $y(l,t)=0, \ \forall t$

(iii) $y(x,0)=0, \ \forall x$.

iv) $\frac{\partial}{\partial t} y(x,0) = \lambda x(l-x)$

The suitable solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow (1)$$

Applying condition (i) in (1) we get,

$$y(0,t)=0$$

$$(A(1)+B(0))(C\cos pat + D\sin pat) = 0.$$

$$A(C\cos pat + D\sin pat) = 0$$

Here $C\cos pat + D\sin pat \neq 0$ [$\because$ It is a function of t]

$$\therefore \boxed{A=0}$$

UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

② $\Rightarrow$ $y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right] \rightarrow$ ③

Applying (iii) in ③,

$$y(x,0) = 0$$

$$B\sin\frac{n\pi x}{l}\left[C(1) + D(0)\right] = 0.$$

$$BC\sin\frac{n\pi x}{l} = 0.$$

Here $B \neq 0$, $\sin\frac{n\pi x}{l} \neq 0 \Rightarrow \boxed{C=0}$

③ $\Rightarrow$ $y(x,t) = B\sin\frac{n\pi x}{l} \cdot D\sin\frac{n\pi at}{l}$

$$= BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= B\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \text{where } B = BD$$

The most general soln is,

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \rightarrow ④$$

Before applying iv) in ④, differentiate p.w.r.t 't'.

$$\frac{\partial}{\partial t}y(x,t) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$\sin 3A = 3\sin A - 4\sin^3 A$
$4\sin^3 A = 3\sin A - \sin 3A$

Applying condition iv) $t=0$

$$\frac{\partial}{\partial t}y(x,0) = V_0\sin^3\frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} B_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l} = \frac{V_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$$B_1\frac{\pi a}{l}\sin\frac{\pi x}{l} + B_2\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + B_3\frac{3\pi a}{l}\sin\frac{3\pi x}{l} + \cdots$$

$$= \frac{3V_0}{4}\sin\frac{\pi x}{l} - \frac{V_0}{4}\sin\frac{3\pi x}{l}$$

UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

Before applying cond (iv), differentiate 'y' w.r.t 't' we get

$$\frac{\partial}{\partial t} y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}\left(\frac{n\pi a}{l}\right)$$

Applying cond (iv)

$$\frac{\partial}{\partial t} y(x,0) = \lambda x(l-x)$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right) = \lambda(lx - x^2)$$

Halfrange sine series,

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \lambda(lx - x^2) \quad \text{where } b_n = B_n\left(\frac{n\pi a}{l}\right)$$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \lambda(lx - x^2) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) - (l-2x)\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{-l}{n\pi}(lx - x^2)\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

$$= \frac{2\lambda}{l}\left[\left(0+0-\frac{2l^3}{n^3\pi^3}\cos n\pi\right)-\left(0+0-\frac{2l^3}{n^3\pi^3}\cos 0\right)\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n+\frac{2l^3}{n^3\pi^3}\right]$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n\frac{n\pi a}{l} = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{4\lambda l^3}{n^4\pi^4 a}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} \frac{8\lambda l^3}{n^4\pi^4 a}, & \text{if } n \text{ is odd.} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(5) \Rightarrow \quad y(x,t) = \sum_{n=odd}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \sum_{n=odd}^{\infty}\frac{8\lambda l^3}{n^4\pi^4 a}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{\pi^4 a}\sum_{n=odd}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

2. If a string of length 'l' is initially at rest in its equilibrium position and each of its points is given by the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = V_0 \sin^3 \frac{\pi x}{l}$, $0 < x < l$. Determine the displacement function $y(x,t)$.

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

the boundary conditions are,

i) $y(0,t) = 0$, $\forall t$

ii) $y(l,t) = 0$, $\forall t$

iii) $y(x,0) = 0$

iv) $\frac{\partial}{\partial t} y(x,0) = V_0 \sin^3 \frac{\pi x}{l}$

The suitable solution is,

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \rightarrow (1)$$

Applying (i) in (1) we get, $y(0,t) = 0$

$(A(1) + B(0))(C\cos\lambda at + D\sin\lambda at) = 0$

Here $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ It is a fn of time)

$\Rightarrow \boxed{A = 0}$

$(1) \Rightarrow y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \rightarrow (2)$

Applying ii) in (2) we get,

$y(l,t) = 0$

$B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0$

Here $B \neq 0$, $C\cos\lambda at + D\sin\lambda at \neq 0$

$\Rightarrow \sin\lambda l = 0 \Rightarrow \lambda l = \sin^{-1} 0$

$\lambda l = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{l}}$

UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

$① \Rightarrow y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow ②$.

Applying condition (ii) in ②, we get,

$y(l,t)=0$

$B\sin pl\,(C\cos pat + D\sin pat) = 0$

Here $C\cos pat + D\sin pat \neq 0$ [∵ It is a function of time]

and $B \neq 0$ [if $B=0$, then we get trivial soln.

$\Rightarrow \sin pl = 0$

$pl = \sin^{-1}(0) \Rightarrow pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

$② \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow ③$

Applying condition (iii) in ③

$y(x,0) = 0$

$B\sin\frac{n\pi x}{l}\left(C(1) + D(0)\right) = 0.$

$BC\sin\frac{n\pi x}{l} = 0$

Here $B \neq 0$ & $\sin\frac{n\pi x}{l} \neq 0$ $\quad \therefore \boxed{C = 0}$

$③ \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left[0 + D\sin\frac{n\pi at}{l}\right]$

$= BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

$= B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

The most general soln is

$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \rightarrow ④$

UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
SOLUTIONS OF ONE DIMENSIONAL WAVE EQUATION

Equating the like coefficients, we get,

$$B_1 \frac{\pi a}{l} = \frac{3V_0}{4} \; ; \; B_2 \cdot \frac{2\pi a}{l} = 0 \; ; \; B_3 \frac{3\pi a}{l} = -\frac{V_0}{4} \; ; \; B_4 \frac{4\pi a}{l} = 0.$$

$$B_1 = \frac{3V_0 l}{4\pi a} \; ; \; B_2 = 0 \; ; \; B_3 = \frac{-V_0 l}{12\pi a} \; ; B_4 = B_5 = 0 \ldots = 0.$$

Subs the above values in (4)

$$y(x,t) = \frac{3V_0 l}{4\pi a} \sin \frac{\pi x}{l} \sin \frac{\pi a t}{l} + 0 - \frac{V_0 l}{12\pi a} \sin \frac{3\pi x}{l} \sin \frac{3\pi a t}{l} + 0$$

$$y(x,t) = \frac{3V_0 l}{4\pi a} \sin \frac{\pi x}{l} \sin \frac{\pi a t}{l} - \frac{V_0 l}{12\pi a} \sin \frac{3\pi x}{l} \sin \frac{3\pi a t}{l}$$