UNIT 3 APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
CLASSIFICATION OF PDE

Introduction : Nature of PDE

The 2nd order PDE is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f(x, y, u, u_x, u_y) = 0 \rightarrow ①$$

(or)

$$A u_{xx} + B u_{xy} + C u_{yy} + f(x, y, u, u_x, u_y) = 0.$$

① is classified as

i) $B^2 - 4AC < 0 \rightarrow$ Elliptic equation

ii) $B^2 - 4AC = 0 \rightarrow$ parabolic equation

iii) $B^2 - 4AC > 0 \rightarrow$ hyperbolic equation.

Problems :

1. Classify the equation $3 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 6 \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial u}{\partial x} - u = 0$

Here $A = 3, B = 4, C = 6$

$$B^2 - 4AC = 16 - 4(3 \times 6) = 16 - 72$$
$$= -56 < 0$$

$\Rightarrow$ It is elliptic equation.

2. Find the nature of the PDE $4u_{xx} + 4u_{xy} + u_{yy} - 2u_x - u_y = 0$

$A = 4, B = 4, C = 1$

$B^2 - 4AC = 16 - 4(4)(1) = 16 - 16 = 0.$

$\Rightarrow$ It is parabolic.

3. Classify the Nature of PDE $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

Here $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow A = 1, B = 0, C = -1$

$B^2 - 4AC = 0 - 4(1)(+1) = 4 > 0$

$\Rightarrow$ It is hyperbolic

4. Classify the PDE $x^2 f_{xx} + (1 - y^2) f_{yy} = 0.$,

$-1 < y < 1$, $-\infty < x < \infty$

$A = x^2$, $B = 0$, $C = 1 - y^2$

$B^2 - 4AC = 0 - 4(x^2)(1 - y^2)$

$\qquad = 4x^2(y^2 - 1)$

In $-\infty < x < \infty$, x^2 is always +ve

In $-1 < y < 1$ $(y^2 - 1)$ is -ve.

$\therefore B^2 - 4AC = 4 (+ve)(-ve) = -ve < 0.$

$\therefore$ It is elliptic

5. Classify $4u_{xx} = u_y$

$\qquad 4u_{xx} - u_y = 0 \Rightarrow A = 4, B = 0, C = 0.$

$\qquad B^2 - 4AC = 0 - 4(4)(0) = 0$

$\qquad \Rightarrow$ It is parabolic