Unit - II

Combinatorics

Principle of Mathematical Induction:

Let $P(n)$ be a statement or proposition involving for all positive integers n.

Step 1: $P(1)$ is true.

Step 2: Assume that $P(K)$ is true

Step 3: we have to prove that $P(K+1)$ is true.

Problems:

1. prove that $2+2^2+\dots+2^n = 2^{n+1}-2$ by mathematical induction.

$P(n): 2+2^2+\dots+2^n = 2^{n+1}-2$

Step 1: $P(1) : 2^1 = 2^{1+1}-2$

$= 2^2-2$

$= 4-2$

$2 = 2$

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(K): 2^1+2^2+\dots+2^K = 2^{K+1}-2$ is true.

Step 3: To prove $P(K+1)$ is true.

Now

$P(K+1) = 2^1+2^2+\dots+2^K+2^{K+1}$

$= 2^{K+1}-2+2^{K+1}$

$= 2^1 . 2^{K+1}-2$

$= 2^{(K+1)+1}-2$

$\therefore$ $P(K+1)$ is true.

$\therefore$ $P(n): 2+2^2+\dots+2^n = 2^{n+1}-2$ is true, for all n.

2). prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ by mathematical Induction

Let $P(n): 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: $P(1)$:

LHS : $1^2 = 1$

RHS : $\frac{1(1+1)(2+1)}{6} = \frac{2(3)}{6} = 1$

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(k): 1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

Step 3: To prove $P(k+1)$ is true.

Now, $P(k+1) = 1^2+2^2+3^2+\dots+k^2+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= (k+1)\frac{[2k^2+k+6k+6]}{6}$

$= (k+1)\frac{[2k^2+7k+6]}{6}$

$= (k+1)\frac{[2k^2+4k+3k+6]}{6}$

$= \frac{(k+1)}{6}[2k(k+2)+3(k+2)]$

$= \frac{(k+1)}{6}[(k+2)(2k+3)]$

$= \frac{(k+1)((k+1)+1)(2k+2+1)}{6}$

$$= \frac{(k+1)\ [(k+1)+1]\ [2(k+1)+1]}{6}$$

$\therefore$ $P(k+1)$ is true.

Hence $P(n): \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ is true, $\forall n$.

3] Prove that $1+2+\dots+n = \frac{n(n+1)}{2}$

Let $P(n): 1+2+\dots+n = \frac{n(n+1)}{2}$

Step 1:

$P(1)$: $LHS = 1$

$RHS = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(k): 1+2+\dots+k = \frac{k(k+1)}{2}$ is true

Step 3: To prove $P(k+1)$ is true.

Now, $P(k+1) = 1+2+\dots+k+k+1$

$$= \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2}$$

$\therefore$ $P(k+1)$ is true.

$\therefore$ $P(n): 1+2+\dots+n = \frac{n(n+1)}{2}$ is true, for all n.

4] Show that $8^n - 3^n$ is a multiple of 5.

Let $P(n): 8^n - 3^n$ be a multiple of 5.

Step 1: $P(1)$: $8^1 - 3^1 = 5$ is a multiple of 5

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(K)$: $8^K - 3^K$ is a multiple of 5, is true.

ie, $8^K - 3^K = 5m \Rightarrow 8^K = 3^K + 5m$

Step 3: To prove $P(K+1)$ is true.

Now $P(K+1) = 8^{K+1} - 3^{K+1}$

$= 8^K \cdot 8^1 - 3^K \cdot 3^1$

$= (3^K + 5m) \cdot 8 - 3^K \cdot 3$

$= 5m \cdot 8 + 3^K \cdot 8 - 3^K \cdot 3$

$= 5m \cdot 8 + 3^K (8-3)$

$= 5m \cdot 8 + 3^K \cdot 5$

$= 5[8m + 3^K]$ which is a multiple of 5, $\forall n$

$\therefore$ $P(K+1)$ is true.

$\therefore$ $P(n)$: $8^n - 3^n$ is a multiple of 5, $\forall n$

5]. Use mathematical induction, Prove that $3^n + 7^n - 2$ is divisible by 8, for $n \geq 1$

Let $P(n)$: $3^n + 7^n - 2$ be divisible by 8, for $n \geq 1$

Step 1: $P(1)$: $3^1 + 7^1 - 2 = 3 + 7 - 2$

$= 8$ is divisible by 8, $n \geq 1$

Step 2: Assume that

$P(K)$: $3^K + 7^K - 2$ is divisible by 8, $n \geq 1$ is true.

ie., $P(K) = \dfrac{3^K + 7^K - 2}{8} = m$

$\Rightarrow 3^K + 7^K - 2 = 8m$

$\Rightarrow 3^K = 8m + 2 - 7^K$

Step 3: To prove $P(K+1)$ is true.

Scanned with CamScanner

Now $P(K+1) = 3^{K+1} + 7^{K+1} - 2$

$$= 3^K \cdot 3 + 7^K \cdot 7 - 2$$
$$= (8m + 2 - 7^K) \cdot 3 + 7^K \cdot 7 - 2$$
$$= 24m + 6 - 3 \cdot 7^K + 7^K \cdot 7 - 2$$
$$= 24m + 4 + 7^K(7-3)$$
$$= 24m + 4 + 7^K \cdot 4$$
$$= 4(6m + 1 + 7^K), \quad m \geq 1, K \geq 1$$

$\because 6m+1+7^K$ is an even number.

$\Rightarrow 4(6m+1+7^K)$ is divisible by 8.

$\therefore P(K+1)$ is true.

$\therefore P(n) = 3^n + 7^n - 2$ is divisible by 8, $n \geq 1$.

6) Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let $P(n): \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1: $P(1)$

LHS: $\frac{1}{1\cdot 2} = \frac{1}{2}$

RHS: $\frac{1}{1+1} = \frac{1}{2}$

Step 2: Assume that

$P(K): \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{K(K+1)} = \frac{K}{K+1}$

Step 3: To prove $P(K+1)$ is true.

Now, $P(K+1) = \frac{1}{1\cdot 2} + \dots + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)}$

$$= \frac{K}{K+1} + \frac{1}{(K+1)(K+2)}$$
$$= \frac{K(K+2)+1}{(K+1)(K+2)} = \frac{K^2+2K+1}{(K+1)(K+2)}$$
$$= \frac{(K+1)(K+1)}{(K+1)(K+2)} = \frac{K+1}{(K+1)+1}$$ is true, $\forall n$