STEADY STATE CONDITIONS & NON-ZERO BOUNDARY CONDITIONS:

A bar 10 cm long with insulated sides has its ends A & B kept at 20°C & 40°C until steady state conditions prevail. The temp at A is then suddenly raised to 50°C and at the same instant that of B is lowered to 10°C. Find the subsequent temp. at any point of the bar at any time.

Sol: The general form of heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

At steady state, $\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$

$\therefore$ The general equation is $u(x) = ax + b$ — (1)

(i) $u(0) = 20$

(ii) $u(10) = 40$

Now $u(x) = ax + b$ (1)

Sub (i) in (1) $u(0) = a(0) + b$

$\boxed{20 = b}$

$\therefore u(x) = ax + 20$ — (2)

Sub (ii) in (2), $u(10) = 10a + 20$

$40 = 10a + 20$

$\boxed{2 = a}$

$\therefore u(x) = 2x + 20$

(i) $u(0) = 50$

(ii) $u(10) = 10$

Now sub (i) in (1)

$u(0) = a(0) + b$

$\boxed{50 = b}$

$\therefore u(x) = ax + 50$ — (2)

Sub (ii) in (2), $u(10) = 10a + 50$

$10 = 10a + 50$

$\boxed{-4 = a}$

$\therefore u(x) = -4x + 50$

Since A is raised to 50°C & B is lowered to 10°C. the steady state is changed to unsteady state.

For unsteady state,

(i) $u(0,t)=50$

(ii) $u(10,t)=10$

(iii) $u(x,0)=f(x)=2x+20$.

Suitable soln is $u(x,t)=u_t(x)+(A\cos px+B\sin px)e^{-\alpha^2p^2t}$ ——— (1)

Sub B.C (i) in (1)

$u(0,t)=-4(0)+50+(A\cos p(0)+B\sin p(0))e^{-\alpha^2p^2t}$

$50=-4(0)+50+Ae^{-\alpha^2p^2t}$

$0=Ae^{-\alpha^2p^2t}$ $[\because e^{-\alpha^2p^2t}\neq 0]$

$\boxed{0=A}$

$\therefore u(x,t)=u_t(x)+B\sin px\,e^{-\alpha^2p^2t}$ ——— (2)

$u(x,t)=-4x+50+B\sin px\,e^{-\alpha^2p^2t}$

Sub. B.C (ii) in (2),

$u(10,t)=u_t(x)+B\sin p(10)\,e^{-\alpha^2p^2t}$

$10=-4(10)+50+B\sin p(10)e^{-\alpha^2p^2t}$

$10=-40+50+B\sin p(10)e^{-\alpha^2p^2t}$

$0=B\sin p(10)e^{-\alpha^2p^2t}$

$[\because e^{-\alpha^2p^2t}\neq 0,\ B\neq 0]$ $\sin p(10)=0$

$\boxed{p=\frac{n\pi}{10}}$

Sub $p=\frac{n\pi}{10}$ in (2), $u(x,t) = u_t(x) + B\sin\left(\frac{n\pi}{10}\right)x\, e^{-\alpha^2\frac{n^2\pi^2}{100}t}$

Steady state soln., $u(x,t) = -4x+50+B\sin\left(\frac{n\pi}{10}\right)x\, e^{-\alpha^2\frac{n^2\pi^2}{100}t}$ ——— (3)

Sub B.C (iii) in (3), $u(x,t) = -4x+50+\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}\right)x\, e^{-\alpha^2\frac{n^2\pi^2}{100}t}$

$u(x,0) = -4x+50+\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}x\right) e^{-\alpha^2\frac{n^2\pi^2}{100}(0)}$

$2x+20 = -4x+50+\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}x\right)$

$6x-30 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{10}\right)x.$

To find B_n, $B_n = \frac{2}{10}\int_0^{10}(6x-30)\sin\left(\frac{n\pi}{10}x\right)dx.$

$u = 6x-30$, $v = \sin\left(\frac{n\pi}{10}\right)x.$

$u' = 6$, $v_1 = -\frac{\cos\left(\frac{n\pi}{10}\right)x}{\frac{n\pi}{10}}$

$u'' = 0$, $v_2 = -\frac{\sin\left(\frac{n\pi}{10}\right)x}{\left(\frac{n\pi}{10}\right)^2}$

$B_n = \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\left(\frac{n\pi}{10}\right)x}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\left(\frac{n\pi}{10}\right)x}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$

$B_n = -\frac{60}{n\pi}\left[1+(-1)^n\right]$

$\therefore u(x,t) = -4x+50+\sum_{n=1}^{\infty}\left[-\frac{60}{n\pi}\left(1+(-1)^n\right)\right]\sin\left(\frac{n\pi}{10}\right)x\, e^{-\alpha^2\frac{n^2\pi^2}{100}t}$