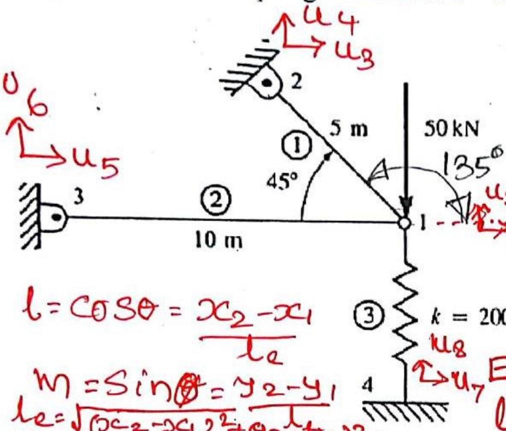
DEPARTMENT OF AEROSPACE ENGINEERING

To illustrate how we can combine spring and bar elements in one structure, we now solve the two-bar truss supported by a spring shown in Figure .1. Both bars have E= 210 GPa and $A = 5\times10^4 m^2$. Bar one has a length of 5 m and bar two a length of 10 m. The spring stiffness is k = 2000 kN/m.



$l = \cos\theta = \dfrac{x_2 - x_1}{l_e}$

$m = \sin\theta = \dfrac{y_2 - y_1}{l_e}$

$l_e = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

Two-bar truss with spring support

Solution :

Stiffness matrix

$$K = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Element ① $\theta = 135°$

$l = \cos 135 = -707.10678\times10^{-3}$

$l^2 = 0.5$

$m = \sin 135 = +707.10678\times10^{-3}$

$m^2 = 0.5$ $[lm = -0.5]$

Stiffness matrix for Element ①

L = 5 m [node (1-2)]

$$K^{(1)} = \frac{(5\times10^{-4})\times(210\times10^9)}{5}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$= 105\times10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

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Element ②, $\theta = 180°$

$l^2 = 1 \quad lm = 0 \quad m^2 = 0$

Stiffness matrix for Element ② node [1-3]

$L = 10\,m$

$$K^{(2)} = \frac{(5\times10^{-4})\times(210\times10^{9})}{10}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 105\times10^{5} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

Element ③, $\theta = 270°$

$l^2 = 0, \quad lm = 0 \quad m^2 = 1$

Stiffness matrix for Element ③ node [1-4] $= K[\;]$

$$K^{(3)} = 20\times10^{5} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}$$

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General Finite Element equation: $[K]\{u\}=\{F\}$

$$10^5\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccccccc} 105+105 & -105 & -105 & 105 & -105 & 0 & 0 & 0 \\ -105 & 105+20 & 105 & -105 & 0 & 0 & 0 & -20 \\ -105 & 105 & 105 & -105 & 0 & 0 & 0 & 0 \\ 105 & -105 & -105 & 105 & 0 & 0 & 0 & 0 \\ -105 & 0 & 0 & 0 & +105 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 & 0 & 0 & 0 & 20 \end{array}\right] \end{array} \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \rightarrow 0 \\ u_4 \rightarrow 0 \\ u_5 \rightarrow 0 \\ u_6 \rightarrow 0 \\ u_7 \rightarrow 0 \\ u_8 \rightarrow 0 \end{bmatrix} = \begin{bmatrix} F_1(0) \\ F_2(-50) \\ F_3(0) \\ F_4(0) \\ F_5(0) \\ F_6(0) \\ F_7(0) \\ F_8(0) \end{bmatrix}$$

Boundary conditions are

$u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$

The final matrix

$$10^5\begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50\times10^3 \end{Bmatrix}$$

$$210\times10^5 u_1 - 105\times10^5 u_2 = 0 \quad \text{--- (1)}$$

$$-105\times10^5 u_1 + 125\times10^5 u_2 = -50\times10^3 \quad \text{--- (2)}$$

$u_1 = -3.448\times10^{-3}\,m$

$u_2 = -6.896\times10^{-3}\,m$

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Stress : Stress of element (1)

$$\sigma^{(1)} = \frac{E}{L_{(1)}} \left[-l_1 \quad -m_1 \quad l_1 \quad m_1 \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\sigma^{(1)}_{(1-2)} = \frac{210\times10^9}{5} \left[0.707 \quad -0.707 \quad -0.707 \quad 0.707 \right] \begin{Bmatrix} -3.448\times10^{-3} \\ -6.896\times10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$= 102.4\,\text{MPa}\,[T]$$

$$\sigma^{(2)}_{(1-3)} = \frac{E}{L_2} \left[-l_2 \quad -m_2 \quad l_2 \quad m_2 \right] \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{bmatrix}$$

$$= \frac{210\times10^9}{10} \left[1 \quad 0 \quad -1 \quad 0 \right] \begin{bmatrix} -3.448\times10^{-3} \\ -6.896\times10^{-3} \\ 0 \\ 0 \end{bmatrix}$$

$$= -72\,\text{MPa}\,[C]$$

$\sigma = P/A$

$f_{1-2} = 102.4\,\text{MPa} \times 10^6 \times 5\times10^{-4}$
$= 51.2\,\text{kN}$

$f_{1-3} = -72.4\,\text{MPa} \times 10^6 \times 5\times10^{-4}$
$= 35.6\,\text{kN}$

Note : can show equilibrium at node 1

$F_s = \left[2000\,\text{kN/m}\right] \times \left[6.896\times10^{-3}\,\text{m}\right] = 13.792\,\text{kN}$

$f_{1-3} = 35.6\,\text{kN}$ | $\Sigma F_y = 0$

$f_{1-2} = 51.2\,\text{kN}$ | $-50 + 13.79 + 36.198 = 0$