Normal Subgroup:

Let H be a subgroup of G under $*$. Then H is said to be a normal subgroup of G, if for every $x \in G$ and for $h \in H$ if

$$x * h * x^{-1} \in H$$

and $x * H * x^{-1} \subseteq H$

Alternatively, a subgroup H of G is called a normal subgroup of G if $x * h = h * x$, $\forall x \in G$.

Theorem: 1

The intersection of any 2 normal subgroups is a normal subgroup.

Proof:

Let H and K be the 2 normal subgroups.

$\Rightarrow H$ and K are subgroups of G.

$\Rightarrow H \cap K$ is a subgroup of G (Already proved)

Now we've to prove that $H \cap K$ is normal.

Let $x \in G$ and $h \in H \cap K$

$x \in G$ and $h \in H$ and $h \in K$

$\Rightarrow x \in G$, and $h \in H$ and $x \in G$, $h \in K$

$\Rightarrow x * h * x^{-1} \in H$ and $x * h * x^{-1} \in K$

$\hookrightarrow (1)$ $\hookrightarrow (2)$ ($\because H$ and K are normal subgroups)

From (1) and (2), we get

$$x * h * x^{-1} \in H \cap K$$

$\Rightarrow H \cap K$ is a normal subgroup of G.

Theorem 2:

Let G and G' be any two groups with identity elt: e and e' resly. If $f: G \to G'$ be a homomorphism, then $\ker(f)$ is a normal subgroup.

Proof:

Let $K = \ker f = \{x \in G / f(x) = e'\}$

we know that $\ker(f)$ is a subgroup of G

Now we've to prove that $\ker(f)$ is normal.

To prove $x * h * x^{-1} \in K$

Let $x \in G$ and $h \in K$

$\therefore f(x * h * x^{-1}) = f(x) * f(h) * f(x^{-1})$ $\because f$ is a homo.

$= f(x) * e' * f(x^{-1})$ $[\because h \in K = \ker f]$

$= f(x) * f(x^{-1})$

$= f(x * x^{-1})$

$= f(e)$

$= e'$

$f(x * h * x^{-1}) = e'$

$\Rightarrow x * h * x^{-1} \in K$

$\therefore K = \ker f$ is a normal subgroup of G.

Theorem: 3 Fundamental theorem of Homomorphism

Every homomorphic image of a group G is isomorphic to some quotient group of G.

(or)

Let $f: G \to G'$ be a onto homomorphism of groups with kernel K. Then $G/K \simeq G'$.

Proof:

Let $f: G \to G'$ be a homomorphism

Let K be the kernel of this homo.

Clearly K is a normal subgroup of G.

To prove G/K is isomorphic $G/K \simeq G'$.

i). To prove ϕ is well defined.

Let $\phi: G/K \to G'$ by $\phi(K*a) = f(a)$.

Consider,

$$K*a = K*b$$
$$\Rightarrow a*b^{-1} \in K$$
$$\Rightarrow f(a*b^{-1}) = e'$$
$$f(a) * f(b^{-1}) = e'$$
$$f(a) * [f(b)]^{-1} = e'$$
$$f(a) * [f(b)]^{-1} * f(b) = e' * f(b)$$
$$f(a) * e = e' * f(b)$$
$$f(a) = f(b)$$
$$\phi(K*a) = \phi(K*b)$$

$\therefore \phi$ is well defined

ii). To prove ϕ is 1-1.

i.e., $\phi(K*a) = \phi(K*b) \Rightarrow K*a = K*b$

Consider $\phi(K*a) = \phi(K*b)$

$f(a) = f(b)$

$f(a) * f(b^{-1}) = f(b) * f(b^{-1})$

$f(a * b^{-1}) = f(b * b^{-1})$

$= f(e)$

$= e'$

$\Rightarrow a * b^{-1} \in K$

$K * a = K * b$

$\therefore \phi$ is 1–1.

(iii). To prove ϕ is onto.

Let $b \in G'$

Since f is onto. $\exists$ an elt. $a \in G$ such that $f(a) = b$.

$\Rightarrow f(a) = \phi(K * a) = b$

$\therefore \phi$ is onto.

(iv). To prove ϕ is a homo.

Now,

$\phi(K * a * K * b) = \phi(K * a * b)$

$= f(a * b)$

$= f(a) * f(b)$

$= \phi(K * a) * \phi(K * b)$

$\therefore \phi$ is a homo.

Since ϕ is 1–1 & onto, homo.

$\therefore \phi$ is an isomorphism b/w G/K and G'

$\Rightarrow G/K \simeq G'$.