FOURIER TRANSFORM :-

Periodic signals can be effectively analysed with the help of fourier series. Fourier transform can be applied for both periodic and non-periodic signals.

Fourier transform is given as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

(or)

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$F[x(t)] = X(\omega) \text{ (or) } X(F)$$

Inverse Fourier Transform :-

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(F)\, e^{+j2\pi ft}\, dF$$

(or)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

If $f(x)$ is the complex function of frequency.

$$X(F) = |X(F)|\, e^{\theta(F)}$$

$$|X(-F)| = |X(F)|$$

$$\theta(-F) = -\theta(F)$$

* Magnitude spectrum is an Even function
* Space spectrum is an odd function

Existence of Fourier transform :-

* The function $x(t)$ should be singled value in any finite time interval. T.

* $x(t)$ should have the most finite no of discontinuities in any time interval T.

* $x(t)$ should have the finite no of maxima and minima in any finite time interval T.

* The function $x(t)$ should be absolutely integrable.

Properties of Fourier Transform :-

1) Linearity :- $x_1(t) \xleftrightarrow{FT} X_1(F)$ & $x_2(t) \xleftrightarrow{FT} X_2(F)$ then

$$c_1 x_1(t) + c_2 x_2(t) \xleftrightarrow{FT} c_1 X_1(F) + c_2 X_2(F)$$

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} c_1 x_1(t) + c_2 x_2(t)\ e^{-j2\pi ft}\, dt$$

$$= c_1 \underbrace{\int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi ft}\, dt} + c_2 \underbrace{\int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi ft}\, dt}$$

$$= c_1 X_1(F) + c_2 X_2(F)$$

2) Time shifting :-

$x(t) \xleftrightarrow{FT} X(F)$ then $x(t-t_0) \leftrightarrow X(F)\, e^{-j2\pi f t_0}$

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi ft}\, dt$$

$t - t_0 = m$, $t = m + t_0$, $dt = dm$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j2\pi f(m+t_0)}\, dm$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j2\pi fm}\, e^{-j2\pi f t_0}\, dm$$

$$= \underbrace{\int_{-\infty}^{\infty} x(m)\, e^{-j2\pi fm}\, dm}\ e^{-j2\pi f t_0}$$

$$= e^{-j2\pi f t_0}\, X(F)$$

3) Frequency shift property :-

If $x(t) \leftrightarrow X(F)$ then $e^{j2\pi f_c t}\, x(t) \leftrightarrow X(F-F_c)$

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{+j2\pi f_c t}\, x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi (F-F_c)t}\, dt$$

$$= X(F-F_c)$$

4) Area Under $x(t)$:-

If $x(t) \leftrightarrow X(F)$ then area under $x(t) = \int_{-\infty}^{\infty} x(t)\, dt = X(0)$

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$F = 0$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

5) Area Under $X(F)$:-

If $x(t) \leftrightarrow X(F)$ then area under $X(F) = \int_{-\infty}^{\infty} X(F)\, dF = x(0)$

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{+j2\pi F t}\, dF$$

$t = 0$

$$x(0) = \int_{-\infty}^{\infty} X(F)\, dF$$

6) Time Scaling :-

$x(t) \leftrightarrow X(F)$ then $x(at) = \frac{1}{|a|} X(F/a)$

a is the scaling factor

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-j2\pi f t}\, dt$$

(i) $a>0$

$at = m, \quad t = m/a, \quad dt = dm/a$

$$X(F) = \int_{-\infty}^{\infty} x(at)\, e^{-j2\pi Ft}\, dt$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j2\pi F(m/a)}\, dm/a$$

$$= 1/a \int_{-\infty}^{\infty} x(m)\, e^{-j2\pi (F/a)m}\, dm$$

$$= 1/a \; X(F/a) \rightarrow (1)$$

(ii) $a<0$ $\quad X(F) = \frac{1}{-a} X(F/a) \rightarrow (2)$

combining (1) & (2)

$$X(F) = \frac{1}{|a|} X(F/a)$$

7) Duality or Symmetry Property :-

$x(t) \leftrightarrow X(F)$ then $X(t) \leftrightarrow x(-F)$

Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi Ft}\, dF$$

Replace $t = -t$ $\qquad t = F$

$\qquad -t = -F$

$$x(-t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi F(-t)}\, dF$$

$$x(-F) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi Ft}\, dF$$

$$X(t) = x(-F)$$

8) Differentiation in time Domain :-

If $x(t) \leftrightarrow X(F)$ then $\frac{d}{dt} x(t) \leftrightarrow j2\pi F\, X(F)$

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi Ft}\, dF$$

$$\boxed{\frac{d}{dt} x(t) = \int_{-\infty}^{\infty} X(F)\ j2\pi F}\ e^{j2\pi F t}\, dF$$

$$\frac{d}{dt} x(t) = X(F)\ j2\pi f$$

9) Integration in time domain :-

$x(t) \leftrightarrow X(F)$ then $\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j2\pi f} X(F)$

Let

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$F[x(t)] = F\left[\frac{d}{dt}\left[\int_{\infty}^{t} x(\tau)\, d\tau\right]\right]$$

$$F\left[\frac{d}{dt} x(t)\right] = j2\pi f\ X(F)$$

$$F\left[\frac{d}{dt} x(t)\right] = j2\pi f\ F[x(t)]$$

$$X(F) = j2\pi f\ F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\frac{1}{j2\pi f} X(F) = F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j2\pi f} X(F)$$

10) conjugate function :-

$x(t) \leftrightarrow X(F)$ then $x^*(t) \leftrightarrow X^*(-F)$

Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(F)\ e^{j2\pi f t}\, dF$$

$$x^*(t) = \int_{-\infty}^{\infty} X^*(F)\ e^{j2\pi f t}\, dF$$

Replace F by $-F$

$$\boxed{x^*(t) = \int_{-\infty}^{\infty} X^*(-F)} \, e^{j2\pi f} \, dF$$

$$x^*(t) \longleftrightarrow X^*(-F)$$

11) Multiplication Theorem :-

$x_1(t) \leftrightarrow X_1(F)$, $x_2(t) \longleftrightarrow X_2(F)$ then

$$x_1(t) \, x_2(t) \longleftrightarrow X_1(F) * X_2(F).$$

Definition :-

$$X_1(F) * X_2(F) = \int_{-\infty}^{\infty} X_1(\lambda) \, X_2(F-\lambda) \, d\lambda$$

$$X(F) = \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi ft} \, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t) \, e^{-j2\pi ft} \, dt$$

$$x_2(t) = \int_{-\infty}^{\infty} X_2(F') \, e^{j2\pi f't} \, dF'$$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(F') \, e^{j2\pi f't} \, dF' \, e^{-j2\pi ft} \, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(F') \, e^{-j2\pi t(F-F')} \, dF' \, dt$$

$F-F' = \lambda$, $F' = F-\lambda$, $dF' = d\lambda$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(F-\lambda) \, e^{-j2\pi t\lambda} \, d\lambda \, dt$$

$$= \int_{-\infty}^{\infty} X_2(F-\lambda) \, d\lambda \int_{-\infty}^{\infty} x_1(t) \, e^{-j2\pi\lambda t} \, dt$$

$$= \int_{-\infty}^{\infty} X_1(\lambda) \, X_2(F-\lambda) \, d\lambda$$

$$= X_1(F) * X_2(F)$$

:) convolution :-

$x_1(t) \leftrightarrow X_1(F)$, $x_2(t) \leftrightarrow X_2(F)$ then

$x_1(t) * x_2(t) \leftrightarrow X_1(F)\, X_2(F)$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) * x_2(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, e^{-j2\pi ft}\, e^{-j2\pi f\tau}\, e^{j2\pi f\tau}\, d\tau\, dt$$

$$\boxed{\therefore e^{-j2\pi f\tau}\, e^{j2\pi f\tau} = 1}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi f(t-\tau)}\, dt$$

$t-\tau = m$, $dt = dm$

$$= X_1(F) \int_{-\infty}^{\infty} x_2(m)\, e^{-j2\pi fm}\, dm$$

$$X(F) = X_1(F)\, X_2(F)$$

13) Rayleigh Energy Theorem (or) Parseval's Theorem :-

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$E = \int_{-\infty}^{\infty} |X(F)|^2\, dF$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(F)|^2\, dF$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \, x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} X^*(F) \, e^{j2\pi ft} \, dF \, dt$$

$$= \int_{-\infty}^{\infty} X^*(F) \int_{-\infty}^{\infty} x(t) \, e^{j2\pi ft} \, dF \cdot dt$$

$$= \int_{-\infty}^{\infty} X^*(F) \, X(F)$$

$$= \int_{-\infty}^{\infty} |X(F)|^2 \, dF \, .$$