# LAPLACE TRANSFORM :-

Fourier transform represents continuous time signal interms of complex sinusoidals. (ie) $e^{j\omega t}$

Laplace transform represents continuous time signals interms of complex exponentials (ie) $e^{-st}$.

continuous time systems can also be analysed effectively using Laplace transforms. Laplace transform of impulse response is called System function (or) Transfer function.

Types of laplace transform :-

1) Bilateral Laplace transform
2) Unilateral Laplace transform.

Laplace Transform :-

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$s = \sigma + j\omega$

$\sigma$ = attenuation constant

$\omega$ = complex frequency.

Unilateral Laplace Transform :-

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$$

Inverse Laplace Transform :-

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

# Relationship between Fourier Transform and Laplace Transform

Fourier Transform :-

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad (or)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \longrightarrow ①$$

Laplace Transform :-

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt$$

$S = \sigma + j\omega$

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt \longrightarrow ②$$

comparing eqn ① & ②

Laplace Transform of $x(t)$ is basically a Fourier transform of $x(t)\, e^{-\sigma t}$

$$L[x(t)] = X(S) = F[x(t)\, e^{-\sigma t}]$$

$\sigma = 0$, becomes

$$L[x(t)] = F[x(t)]$$

$$\downarrow$$

$$X(S) = X(j\omega) \quad , \text{ where } S = j\omega$$

$S = j\omega$ indicates Imaginary accessing complex's plane.

Convergence of Laplace Transform :-

Laplace transform is basically a fourier transform of $x(t)\, e^{-\sigma t}$. Fourier transform of $x(t)\, e^{-\sigma t}$ exist, then $L[x(t)]$ also exist.

Fourier transform of $x(t)\, e^{-\sigma t}$ must be absolutely integrable.

$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt < \infty$$

ROC :- [Region of convergence]

The Range of value of $\sigma$ for which the Laplace transform convergence is called region of convergence.

Problems :-

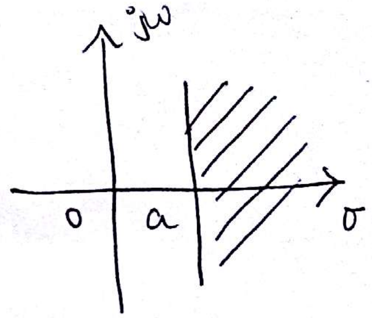
(1) Find the Laplace transform of $x(t) = e^{at} u(t)$ and plot its ROC.

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{at} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{at}\, e^{-st}\, dt$$

$$= \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \Rightarrow \frac{1}{s-a}, \quad s-a>0, \quad s>a.$$



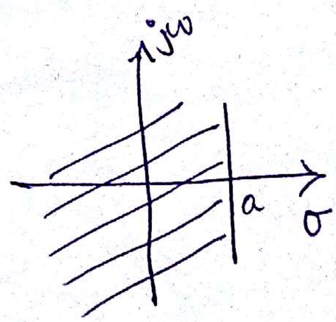

(2) Find the Laplace transform of $x(t) = e^{-at} u(-t)$

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(-t)\, e^{-st}\, dt$$

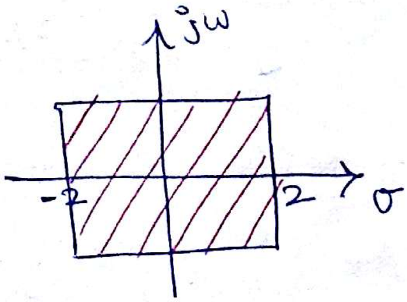
$$= \int_{-\infty}^{0} -e^{-at}\, e^{-st}\, dt$$

$$= \left[ \frac{-e^{-t(s-a)}}{-(s-a)} \right]_{-\infty}^{0} \Rightarrow \frac{1}{s-a}, \quad s-a<0, \quad s<a.$$



③ $x(t) = e^{-2t}u(t) - e^{2t}u(-t)$

$$X(S) = \frac{1}{S+2} - \frac{1}{S-2}$$




$S+2>0$ $\quad$ $S-2>0$

$S>-2$ $\quad$ $S<2$

$-2<S<2$

④ Find Laplace transform of $r(t)$

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$u = t \quad dv = e^{-st}$

$u' = 1 \quad V = \frac{e^{-st}}{-s}$

$V_1 = \frac{e^{-st}}{s^2}$

$$= \int_{-\infty}^{\infty} t\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^{\infty} = \frac{1}{s^2}$$

⑤ Find Laplace Transform of $x(t) = e^{-at}\cos\omega t\, u(t)$

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\cos\omega t\; u(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) u(t)\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-at}\left(e^{j\omega t} + e^{-j\omega t}\right) u(t)\, e^{-st}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-at}\, e^{j\omega t}\, u(t)\, e^{-st}\, dt + \int_{-\infty}^{\infty} e^{-at}\, e^{-j\omega t}\, u(t)\, e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{(j\omega-a)t}\, u(t)\, e^{-st}\, dt + \int_{-\infty}^{\infty} e^{(j\omega+a)t}\, u(t)\, e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[L\left[e^{(+j\omega-a)t}\, u(t)\right] + L\left[e^{(j\omega+a)t}\, u(t)\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega+a} + \frac{1}{s+j\omega+a}\right]$$

$$= \frac{1}{2}\left(\frac{2s+2a}{s^2+\omega^2+a^2}\right) \Rightarrow \frac{1}{\cancel{2}}\left(\frac{\cancel{2}(s+a)}{s^2+\omega^2+a^2}\right)$$

$$X(s) = \frac{s+a}{s^2+a^2+\omega^2}$$

(HW)

6. Find the Laplace Transform of $x(t) = e^{-at} \sin \omega_0 t\, u(t)$

$$X(s) = \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

7. Find the Laplace transform of $x(t) = \sinh at$

$$\left[\sinh at = \frac{e^{at}-e^{-at}}{2}\right]$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \sinh at\; e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{at}-e^{-at}}{2}\right) e^{-st}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{at}-e^{-at}\right) e^{-st}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{at}\, e^{-st}\, dt - \int_{-\infty}^{\infty} e^{-at}\, e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right) = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{s^2-a^2}\right] = \frac{a}{s^2+a^2}$$

(8) Find the Laplace transform of $x(t) = \cosh at$

$$X(s) = \frac{s}{s^2+a^2}$$

(9) Find the Laplace transform of $x(t) = e^{-at}\cosh bt$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} \cosh bt\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\left(\frac{e^{bt}+e^{-bt}}{2}\right) e^{-st}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-at}\left(e^{bt}+e^{-bt}\right) e^{-st}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-at} e^{bt} e^{-st}\, dt + \int_{-\infty}^{\infty} e^{-at} e^{-bt} e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{(b-a)t} e^{-st}\, dt + \int_{-\infty}^{\infty} e^{-(b+a)t} e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[L\left[e^{(b-a)t}\right] + L\left[e^{-(b+a)t}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-b+a} + \frac{1}{s+b+a}\right]$$

$$\therefore = \frac{1}{2}\left[\frac{s+b+a+s-b+a}{s^2-b^2+a^2}\right] \Rightarrow \frac{1}{2}\left[\frac{2s+2a}{s^2-b^2+a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(s+a)}{s^2-b^2+a^2}\right]$$

$$X(s) = \frac{s+a}{(s+a)^2-b^2}$$

HW

10) Find the Laplace transform of $x(t) = e^{-at}\sinh bt$

$$X(s) = \frac{b}{(s+a)^2-b^2}$$

## Unilateral Laplace Transform :-

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

Roc need not be specified with Unilateral Laplace transform.

Inverse Laplace transform using partial fraction Method :-

$$L\left[e^{at}u(t)\right] = \frac{1}{s-a}, \quad s>a$$

$$L\left[-e^{at}u(-t)\right] = \frac{1}{s-a}, \quad s<a$$

$$L\left[e^{-at}u(t)\right] = \frac{1}{s+a}, \quad s>-a$$

$$L\left[-e^{-at}u(-t)\right] = \frac{1}{s+a}, \quad s<-a$$