PROPERTIES OF LAPLACE TRANSFORM

1) Linearity Property :-

Let $x_1(t) \leftrightarrow X_1(s)$ ROC ; R_1 at $x_2(t) \leftrightarrow X_2(s)$ ROC ; R_2

then

$$L[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(s) + a_2 X_2(s)$$

Proof :-

$$L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-st}\, dt$$

$$= a_1 \underbrace{\int_{-\infty}^{\infty} x_1(t)\, e^{-st}\, dt}_{X_1(s)} + a_2 \underbrace{\int_{-\infty}^{\infty} x_2(t)\, e^{-st}\, dt}_{X_2(s)}$$

$$L[x(t)] = a_1 X_1(s) + a_2 X_2(s)$$

2) Time shifting property : [Translation in Time Domain]

Let $x(t) \leftrightarrow X(s)$ ROC : R then

$$L[x(t-t_0)] = e^{-st_0}\, X(s)$$

Proof :-

$$L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$$

$t - t_0 = m$, $t = m + t_0$, $dt = dm$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-s(m+t_0)}\, dm$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-sm}\, e^{-st_0}\, dm$$

$$= e^{-st_0} \underbrace{\int_{-\infty}^{\infty} x(m)\, e^{-sm}\, dm}_{X(s)}$$

$$= e^{-st_0}\, X(s)$$

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(3) Translation in frequency domain :-

Let $x(t) \leftrightarrow X(S)$ ROC; R then

$$L\left[e^{S_0 t} x(t)\right] = X(S - S_0)$$

Proof :-

$$L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt$$

$$= \int_{-\infty}^{\infty} e^{S_0 t} x(t)\, e^{-St}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-t(S - S_0)}\, dt}_{X(S - S_0)}$$

$$= X(S - S_0)$$

(4) Time scaling :-

Let $x(t) \leftrightarrow X(S)$ ROC; R then $x(at) \leftrightarrow \frac{1}{|a|} X(S/a)$

Proof :-

$$L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt$$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-St}\, dt$$

case(i) $a > 0$; $at = m$, $t = m/a$, $dt = dm/a$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-S(m/a)}\, \frac{dm}{a}$$

$$= \frac{1}{a} \underbrace{\int_{-\infty}^{\infty} x(m)\, e^{-(S/a)m}\, dm}$$

$$= \frac{1}{a} X(S/a) \rightarrow ①$$

case(ii) $a < 0$;

$$L[x(at)] = \frac{1}{-a} X(S/a) \rightarrow ②$$

By combining ① & ②

$$= \frac{1}{|a|} X(S/a)$$

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5) Differentiation in Time Domain :

Let $x(t) \leftrightarrow X(s)$ ROC ; R then

$$\frac{d}{dt} x(t) \leftrightarrow s X(s)$$

Proof :

Inverse LT : $x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$

$$\frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \frac{d}{dt} e^{st} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) (s) e^{st} ds$$

$$\boxed{\frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} s X(s)} e^{st} ds$$

$$\frac{d}{dt} x(t) \leftrightarrow s X(s)$$

(6) Differentiation in S-domain :

Let $x(t) \leftrightarrow X(s)$ ROC ; R then

$$-t\, x(t) \leftrightarrow \frac{d}{ds} X(s)$$

Proof :

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t) \frac{d}{ds} (e^{-st}) dt$$

$$= \int_{-\infty}^{\infty} x(t) (-t) e^{-st} dt$$

$$\boxed{\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} -t\, x(t)} e^{-st} dt$$

$$\frac{d}{ds} X(s) \leftrightarrow -t\, x(t)$$

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(7) Integration in Time Domain :-

If $x(t) \leftrightarrow X(s)$ ROC ; R then

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{X(s)}{s}$$

Proof :-

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$u(t-\tau) = \{ 1 \text{ for } t \geq \tau, \tau \leq t$$

$$= \int_{-\infty}^{\infty} x(\tau)\, d\tau$$

$$L\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = L\left[x(t) * u(t)\right]$$

$$= X(s) \cdot 1/s$$

$$= \frac{X(s)}{s}$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{X(s)}{s}$$

(8) Integration in s-domain :-

$x(t) \leftrightarrow X(s)$ ROC ; R then $\dfrac{x(t)}{t} \leftrightarrow \displaystyle\int_{s}^{\infty} X(s)\, ds$

Proof :-

$$= \int_{s}^{\infty} X(s)\, ds$$

$$= \int_{s}^{\infty} \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt\, ds$$

$$= \int_{-\infty}^{\infty} x(t)\, dt \int_{s}^{\infty} e^{-st}\, ds$$

$$= \int_{-\infty}^{\infty} x(t)\, dt \left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} \Rightarrow \int_{-\infty}^{\infty} x(t)\, dt \quad \frac{e^{-st}}{t}$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t}\, e^{-st}\, dt$$

$$\int_{s}^{\infty} X(s)\, ds = \int_{-\infty}^{\infty} \frac{x(t)}{t}\, e^{-st}\, dt$$

$$\boxed{\frac{x(t)}{t} \leftrightarrow \int_{s}^{\infty} X(s)\, ds}$$

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9) convolution Property :-

$x_1(t) \leftrightarrow X_1(S)$ ROC ; R_1 and $x_2(t) \leftrightarrow X_2(S)$ ROC ; R_2

then

$$x_1(t) * x_2(t) \leftrightarrow X_1(S) \; X_2(S)$$

Proof :-

$$L[x(t)] = \int_{-\infty}^{\infty} x(t) \, e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] \, e^{-st} \, dt$$

Definition :-

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \, x_2(t-\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \, x_2(t-\tau) \, d\tau \, e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \, d\tau \int_{-\infty}^{\infty} x_2(t-\tau) \, e^{-st} \, dt$$

By time shifting property :-

$$L[x(t-t_0)] = e^{-st_0} \, X(S)$$

$$L[x(t-\tau)] = e^{-s\tau} \, X(S) \Rightarrow L[x_2(t-\tau)] = e^{-s\tau} X_2(S)$$

$$= \underbrace{\int_{-\infty}^{\infty} x_1(\tau) \, e^{-s\tau} \, d\tau}_{X_1(S)} \cdot X_2(S)$$

$$= X_1(S) \cdot X_2(S)$$

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