Period: 6

Eigen Values and Eigen Vectors

Solve the characteristic equation, we get eigen.

If there exist a non-zero vectors x such that $|A - \lambda I|\bar{x} = 0$, then the vector x is called eigen vector.

Q) Find the eigen values and eigen vectors of the matrix.

i) $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

Step 1 :- characteristic equation

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \longrightarrow (1)$$

D_1 = Sum of the main diagonal element

$= 7 + 6 + 5 = 18$

D_2 = Sum of the minors of the main diagonal elements.

$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$

$= [30 - 4) + (35) + (42 - 4)$

$= 26 + 35 + 38 = 99$

$D_3 = |A|$

$= \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$

$7\begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -2 \\ 0 & 5 \end{vmatrix} + 0$

$7(30 - 4) + 2(-10)$

$7(26) + 2(-10)$

$= 182 - 20 = 162$

$\therefore \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

Step :- 2 To find eigen values

$$\begin{array}{c|cccc} 3 & 1 & -18 & 99 & -162 \\ & 0 & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & 0 \end{array}$$

$\lambda = 3$.

$\lambda^2 - 15 + 54 = 0$

$(\lambda - 6)(\lambda - 9) = 0$

$\lambda = 6, 9$

The eigen values are 3, 6, 9.

Step 3 :- To find eigen vectors

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

case 1, when $\lambda = 3$

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{array}{cccc} x_1 & & x_2 & x_3 \\ -2 & 0 & 4 & -2 \\ 3 & -2 & -2 & 3 \end{array}$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

case :- 1

$\lambda = 1$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{pmatrix}$$

$$\begin{array}{cccc} x_1 & & x_2 & & x_3 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{array}$$

$$\frac{x_1}{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} = \frac{x_2}{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}} = \frac{x_3}{\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}}$$

$$\frac{x_1}{(2-0)} = \frac{x_2}{(0-1)} = \frac{x_3}{(0-4)}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4}$$

$$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$$

case :- 2

$\lambda = 3$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{pmatrix}$$

$$\begin{array}{cccc} x_1 & & x_2 & & x_3 \\ 2 & 0 & -1 & 2 \\ -2 & 1 & 2 & -2 \end{array}$$

Eigen values	Eigen vectors
1	$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$
2	$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$
3	$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

Q/ Find the eigen values and eigen vectors of the matrix.

3/ $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ 4) $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Step :- 3 To find the eigen vectors

$$(A - \lambda I)\, X = 0$$

$$\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case :- 1

when $\lambda = 1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{array}{cccccc} & x_1 & & x_2 & x_3 & \\ 0 & & 1 & 1 & & 0 \\ 2 & & 0 & 0 & & 2 \end{array}$$

$$\frac{x_1}{(0-2)} = \frac{x_2}{(0-0)} = \frac{x_3}{(2+0)}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

case :- 2

when $\lambda = 3$,

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 0x_2 + x_3 = 0$$

$$-x_1 + x_3 = 0$$

$$\cancel{+}x_1 = \cancel{+}x_3$$

$$\frac{x_1}{1} = \frac{x_3}{1}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 4 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix}}$$

$$\frac{x_1}{4-0} = \frac{x_2}{0+8} = \frac{x_3}{12-4}$$

$$\frac{x_1}{4} = \frac{x_2}{8} + \frac{x_3}{8} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

case :- 2 when $\lambda = 6$

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication method

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -2 & 0 & 1 & -2 \\ 0 & -2 & -2 & 0 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ -2 & 0 \end{vmatrix}}$$

$$\frac{x_1}{(4-0)} = \frac{x_2}{(0+2)} = \frac{x_3}{(0-4)}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case :- 3 $\lambda = 9$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule,

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -2 & 0 & -2 & -2 \\ -3 & -2 & -2 & -3 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \begin{vmatrix} -2 & 0 \\ -3 & -2 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} -2 & -2 \\ -2 & -3 \end{vmatrix} \end{array}$$

$$\frac{x_1}{(4+0)} = \frac{x_2}{(0-4)} = \frac{x_3}{(6-4)}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Conclusion:-

Period: 3

Eigen values	Eigen vectors
3	$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$
6	$\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$
9	$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Find the eigen values & eigen vectors of matrix.

9, $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$