



Elastic deformation:

Example 1 :-

Stretching of an elastic membrane:

An elastic membrane in the x_1, x_2 plane, with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so. There is a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ is given by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax \text{ where } Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

in components
$$\begin{cases} y_1 = 5x_1 + 3x_2 \\ y_2 = 3x_1 + 5x_2 \end{cases}$$

Then find the principle direction i.e. the direction of the positive vector for which the direction vector of the position vector $y(Q)$ are exactly opposite. What does the boundary circle take under the deformation?

Soln:



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Given $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ find co-ordinate \Rightarrow To find the slope $[x=0, y=0; n= \text{value}]$

The characteristic equation of (2×2) matrix is $\lambda^2 - D_1\lambda + D_2 = 0$. To find principle direction we need the angle \Rightarrow find angle $\Rightarrow \cos \alpha = \frac{x}{\sqrt{x^2+y^2}}, \sin \alpha = \frac{y}{\sqrt{x^2+y^2}}$

$$D_1 = 10,$$

$$D_2 = 16.$$

The characteristic equation is $\lambda^2 - 10\lambda + 16 = 0$

The eigen values are $\lambda = 2, 8$

Eigen vectors:

$$(A - \lambda I)x = 0.$$

$$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{①}$$

When $\lambda = 2$ in ①

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$3x_1 + 3x_2 = 0.$$

$$\Rightarrow 3x_1 + 3x_2 = 0$$

$$3x_1 = -3x_2$$

$$x_1 = -x_2$$

By assumption method.

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

When $\lambda = 8$ in ①.

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 3x_2 = 0$$

$$3x_1 - 3x_2 = 0$$

$$3x_1 = 3x_2$$

$$x_1 = x_2$$

By assumption method

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$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We know that $\cos \alpha = \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}$

$$\text{When } x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \cos \alpha = \frac{-1}{\sqrt{(-1)^2+(1)^2}}$$

$$\alpha = \cos^{-1}(-1/\sqrt{2})$$

$$\alpha = 135^\circ$$

$$\text{When } x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \cos^{-1} = \left[\frac{1}{\sqrt{1^2+1^2}} \right]$$

$$\alpha = \cos^{-1}(1/\sqrt{2})$$

$$\alpha = 45^\circ$$

Polar co-ordinates to find deformation.

The principle direction occurs in the angle 45° and 135° .

$$x = a \cos \theta \quad y = b \sin \theta$$

$$\text{Let } a = 2, b = 8$$

$$x = 2 \cos \theta$$

$$y = 8 \sin \theta$$

$$\cos \theta = x/2 \quad \sin \theta = y/8$$

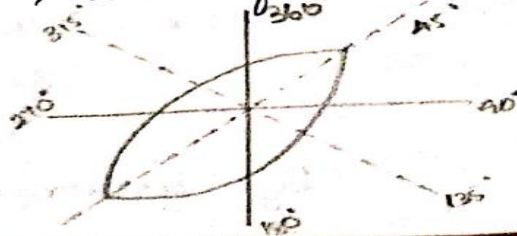
We know that

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$(x/2)^2 + (y/8)^2 = 1$$

$$x^2/2^2 + y^2/8^2 = 1$$

which is ellipse, The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



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