Centre of curvature & Circle of curvature

Centre of curvature at any pt. on the curve $y = f(x)$ is $C(\bar{x}, \bar{y})$

where $$\bar{x} = x - \frac{dy}{dx}\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]$$

$$\bar{y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

(or)

$$\bar{x} = x - \frac{y_1\left[1+y_1^2\right]}{y_2}$$

$$\bar{y} = y + \left[\frac{1+y_1^2}{y_2}\right]$$

Circle of curvature at any point is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$ where ρ is the radius of curvature.

I. Find the centre and circle of curvature at (c, c) on $xy = c^2$.

Soln.

Given. $xy = c^2$

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

At (c, c), $y_1 = -1$

and $\frac{d^2y}{dx^2} = \frac{-[x\,y_1 - y(1)]}{x^2} = \frac{-xy_1 + y}{x^2}$

At (c, c), $y_2 = \frac{-c(-1)+c}{c^2} = \frac{2c}{c^2}$

$y_2 = \frac{2}{c}$

$\therefore \rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$= \frac{[1+(-1)^2]^{3/2}}{2/c}$

$= \frac{2^{3/2}}{2/c} = \frac{c}{2} \cdot 2\sqrt{2}$

$= \sqrt{2}\,c$

To find $\bar{x}$ & $\bar{y}$:

$\bar{x} = x - \frac{y_1[1+y_1^2]}{y_2} = x + \frac{1[1+(-1)^2]}{2/c}$

$= x + 2 \times \frac{c}{2}$

$\bar{x} = x + c$

$\bar{y} = y + \frac{[1+y_1^2]}{y_2} = y + \frac{[1+(-1)^2]}{2/c}$

$= y + 2 \times \frac{c}{2}$

$\bar{y} = y + c$

At (c, c): $\bar{x} = c + c = 2c$

$\bar{y} = c + c = 2c$

centre of curvature $c(\bar{x}, \bar{y}) = c(2c, 2c)$

circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$(x-2c)^2 + (y-2c)^2 = (c\sqrt{2})^2$