Type 1:

Find the envelope of $y = mx + \frac{a}{m}$, m being the parameter.

Soln.

$$y = mx + \frac{a}{m}$$

$$my = m^2x + a$$

$$m^2x - my + a = 0$$ which is the quadratic eqn. with parameter m.

$$Ax^2 + Bx + C = 0 \Rightarrow B^2 - 4AC = 0$$

Now, $A = x$, $B = -y$, $C = a$

$\therefore\ y^2 - 4ax = 0$ which is the envelope.

Find the envelope of $y = mx + \sqrt{a^2m^2 - b^2}$, m is the parameter.

Soln.

Given. $y = mx + \sqrt{a^2m^2 - b^2}$

$$y - mx = \sqrt{a^2m^2 - b^2}$$

Squaring on bothsides,

$$(y - mx)^2 = a^2m^2 - b^2$$

$$y^2 + m^2x^2 - 2mxy = a^2m^2 - b^2$$

$$y^2 + m^2x^2 - 2mxy - a^2m^2 + b^2 = 0$$

$$m^2(x^2 - a^2) - 2xy\,m + (y^2 + b^2) = 0$$

Here $A = x^2 - a^2$, $B = -2xy$, $C = y^2 + b^2$

Now $B^2 - 4AC = 0$

$$4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

$$4x^2y^2 - 4[x^2y^2 + x^2b^2 - a^2y^2 - a^2b^2] = 0$$

$\div 4$ $$x^2y^2 - x^2y^2 - x^2b^2 + a^2y^2 + a^2b^2 = 0$$

$\div(-a^2b^2)$ $$-x^2b^2 + a^2y^2 = -a^2b^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ which is the envelope.

Type 2:

1. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$ subject to $a+b=c$, where c is a constant.

Soln.

Given. $\frac{x}{a}+\frac{y}{b}=1 \rightarrow (1)$

$a+b=c \rightarrow (2)$

Differentiate (1) w.r. to 'a'

$$x\left[-\frac{1}{a^2}\right]+y\left[-\frac{1}{b^2}\right]\frac{db}{da}=0$$

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$$

$$-\frac{y}{b^2}\frac{db}{da}=\frac{x}{a^2}$$

$$\frac{db}{da}=-\frac{xb^2}{ya^2} \rightarrow (3)$$

Differentiate (2) w.r. to a

$$1+\frac{db}{da}=0$$

$$\frac{db}{da}=-1 \rightarrow (4)$$

From (3) and (4),

$$-\frac{xb^2}{ya^2}=-1$$

$$\frac{x}{a^2}=\frac{y}{b^2}$$

$$\frac{\frac{x}{a}}{a}=\frac{y/b}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}$$

$$\therefore \frac{x}{a^2}=\frac{1}{c} \text{ and } \frac{y}{b^2}=\frac{1}{c}$$

$a^2 = xc$ | $b^2 = yc$

$a = (xc)^{1/2}$ | $b = [yc]^{1/2}$

Subs in (2),

$$(xc)^{1/2} + (yc)^{1/2} = c$$

$$c^{1/2}[x^{1/2} + y^{1/2}] = c$$

$$x^{1/2} + y^{1/2} = c^{1/2}$$

2). Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ subject to $a^n + b^n = c^n$, where c is constant

Soln.

Gvn. $\frac{x}{a} + \frac{y}{b} = 1 \rightarrow (1)$

$a^n + b^n = c^n \rightarrow (2)$

Differentiate (1) w.r. to 'a'

$$x\left(-\frac{1}{a^2}\right) + y\left(-\frac{1}{b^2}\right)\frac{db}{da} = 0$$

$$-\frac{y}{b^2}\frac{db}{da} = \frac{x}{a^2}$$

$$\frac{db}{da} = -\frac{xb^2}{ya^2} \rightarrow (3)$$

Differentiate (2) w.r. to 'a'

$$na^{n-1} + nb^{n-1}\frac{db}{da} = 0$$

$$nb^{n-1}\frac{db}{da} = -na^{n-1}$$

$$\frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \rightarrow (4)$$

From (3) and (4),

$$-\frac{xb^2}{ya^2} = -\frac{a^{n-1}}{b^{n-1}}$$

$$\frac{x}{a^{n-1}a^2} = \frac{y}{b^{n-1}b^2}$$

$$\frac{x}{a^{n+1}} = \frac{y}{b^{n+1}}$$

i.e., $$\frac{\frac{x}{a}}{a^n} = \frac{y/b}{b^n} = \frac{x/a + y/b}{a^n+b^n} = \frac{1}{c^n}$$

$$\Rightarrow \frac{x}{a^{n+1}} = \frac{1}{c^n} \qquad \frac{y}{b^{n+1}} = \frac{1}{c^n}$$

$$a^{n+1} = xc^n \qquad b^{n+1} = yc^n$$

$$a^{n+1 \times \frac{n}{n+1}} = (xc^n)^{n/n+1} \qquad b^{(n+1)\times n/(n+1)} = (yc^n)^{n/(n+1)}$$

$$a^n = (xc^n)^{n/n+1} \qquad b^n = (yc^n)^{n/n+1}$$

Subs. in (2)

$$(xc^n)^{n/(n+1)} + (yc^n)^{n/(n+1)} = c^n$$

$$c^{n^2/(n+1)}\left[x^{n/n+1} + y^{n/n+1}\right] = c^n$$

$$x^{n/n+1} + y^{n/n+1} = c^{n - n^2/(n+1)} \qquad \frac{n^2+n-n^2}{n+1}$$

$$x^{n/n+1} + y^{n/n+1} = c^{n/n+1}$$

HW

1. Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ Subject to $a^2+b^2=c^2$ where c is a constant.

2. Find the envelope of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Subject to $a+b=c$, c is a constant.

Soln.

Given. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow (1)$$

$$a+b = c \rightarrow (2)$$

Differentiate (1) w.r. to a

$$x^2\left[\frac{-2}{a^3}\right] + y^2\left[\frac{-2}{b^2}\right]\frac{db}{da} = 0$$

$\div 2$

$$-\frac{y^2}{b^3}\frac{db}{da} = \frac{x^2}{a^3}$$

$$\frac{db}{da} = -\frac{x^2 b^3}{a^3 y^2} \rightarrow (3)$$

Differentiate (2) w.r. to a

$$1 + \frac{db}{da} = 0$$

$$\frac{db}{da} = -1 \rightarrow (4)$$

From (3) & (4),

$$\frac{-x^2 b^3}{a^3 \cdot y^2} = -1$$

$$\frac{x^2}{a^3} = \frac{y^2}{b^3}$$

$$\frac{\frac{x^2}{a^2}}{a} = \frac{\frac{y^2}{b^2}}{b}$$

ie., $\dfrac{\frac{x^2}{a^2}}{a} = \dfrac{\frac{y^2}{b^2}}{b} = \dfrac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \dfrac{1}{c}$

$\Rightarrow \frac{x^2}{a^3} = \frac{1}{c}$ | $\frac{y^2}{b^3} = \frac{1}{c}$

$a^3 = cx^2$ | $b^3 = cy^2$

$a = (cx^2)^{1/3}$ | $b = (cy^2)^{1/3}$

Subs. in (2),

$$(cx^2)^{1/3} + (cy^2)^{1/3} = c$$

$$c^{1/3} x^{2/3} + c^{1/3} y^{2/3} = c$$

$$c^{1/3}[x^{2/3} + y^{2/3}] = c$$

$$x^{2/3} + y^{2/3} = c^1 c^{-1/3}$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

4]. Find the envelope of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ subject to $a^n+b^n=c^n$, c is constant.

Soln.

Gvn. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow (1)$

$a^n+b^n=c^n \rightarrow (2)$

Diff. (1) w.r. to 'a',

$x^2\left(-\frac{2}{a^3}\right)+y^2\left(-\frac{2}{b^3}\right)\frac{db}{da}=0$

$\div 2$ $\quad -\frac{y^2}{b^3}\frac{db}{da}=\frac{x^2}{a^3}$

$\frac{db}{da}=\frac{-x^2 b^3}{y^2 a^3} \rightarrow (3)$

Diff (2) w.r. to 'a',

$na^{n-1}+nb^{n-1}\frac{db}{da}=0$

$\div n$ $\quad b^{n-1}\frac{db}{da}=-a^{n-1}$

$\frac{db}{da}=\frac{-a^{n-1}}{b^{n-1}} \rightarrow (4)$

From (3) & (4), $\frac{-x^2 b^3}{y^2 a^3}=\frac{-a^{n-1}}{b^{n-1}}$

$\frac{x^2}{a^3 a^{n-1}}=\frac{y^2}{b^{n-1}b^3}$

ie., $\frac{x^2}{a^{n+2}}=\frac{y^2}{b^{n+2}}$

$\frac{\frac{x^2}{a^2}}{a^n}=\frac{y^2/b^2}{b^n}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^2+b^n}=\frac{1}{c^n}$

$\Rightarrow \frac{x^2}{a^{n+2}}=\frac{1}{c^n} \quad \Big| \quad \frac{y^2}{b^{n+2}}=\frac{1}{c^n}$

$$\therefore \frac{\frac{x}{a}}{1} = \frac{\frac{y}{b}}{1} = \frac{\frac{x}{a}+\frac{y}{b}}{1+1} = \frac{1}{2}$$

$$\therefore \frac{x}{a} = \frac{1}{2} \qquad \Big| \qquad \frac{y}{b} = \frac{1}{2}$$

$$a = 2x \qquad \Big| \qquad b = 2y$$

Sub. (a) & (b) in (2),

$$(2x)(2y) = c^2$$

$$4xy = c^2$$