Reduce the quadratic form

$Q = x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_1x_3$

into canonical form using orthogonal

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = -2, 3$$

Eigen values are $6, -2, 3$

Eigen vectors

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case :-1 When $\lambda = -2$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 3 & 3 & 1 \\ 7 & 1 & 1 & 7 \end{array}$$

$$\frac{x_1}{(1-21)} = \frac{x_2}{(3-3)} = \frac{x_3}{(21-1)}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case :- 2 when $\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{(1-6)} = \frac{x_2}{(3+2)} = \frac{x_3}{(-4-1)}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

case :- 3 When $\lambda = 6$

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{(1+3)} = \frac{x_2}{(3+5)} = \frac{x_3}{(5-1)}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Step :- 3

$$x_1^T \cdot x_2 = (-1\ 0\ 1)\begin{pmatrix}-1\\1\\-1\end{pmatrix} = 1-1 = 0$$

$$x_2^T \cdot x_3 = (-1\ 1\ -1)\begin{pmatrix}1\\2\\1\end{pmatrix} = -1+2-1 = 0$$

$$x_3^T \cdot x_1 = (1\ 2\ 1)\begin{pmatrix}-1\\0\\1\end{pmatrix} = -1+0+1 = 0$$

∴ Eigen vectors are pairwise orthogonal.

Step :- 4

Normalized Eigen vectors

$$x_1 = \begin{pmatrix}-1\\0\\1\end{pmatrix} \quad l(x) = \sqrt{(-1)^2+(0)^2+(1)^2} = \sqrt{2}$$

$$x_1 = \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$$

$$x_2 = \begin{pmatrix}-1\\1\\-1\end{pmatrix} \quad l(x) = \sqrt{(-1)^2+(1)^2+(-1)^2} = \sqrt{3}$$

$$x_2 = \begin{pmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}$$

$$x_3 = \begin{pmatrix}1\\2\\1\end{pmatrix} \quad l(x) = \sqrt{(1)^2+(2)^2+(1)^2} = \sqrt{6}$$

$$x_3 = \begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$$

Step 5 :-

Normalized model matrix

$$N = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

Step :- 6

N should be orthogonal

i.e, $NN^T = N^T N = I$

Now,

$$NN^T = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

N is orthogonal

Step :- 7 $D = N^T A N$

$$= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Step :- 8 canonical form $(CF) = Y^T D Y$

$$CF = (y_1 \; y_2 \; y_3) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$-2y_1^2 + 3y_2^2 + 6y_3^2$$

→ Rank = 3 → signature = 1

→ Index = 2 → Nature = Indefinite