Elastic deformation:

Example 1 :

Stretching of an elastic membrane:

An elastic membrane in the x_1, x_2 plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretch so. There is a point $P(x_1, x_2)$ goes over into the point $Q(y_1, y_2)$ is given by $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax$ where, $Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

in components $\begin{bmatrix} y_1 = 5x_1 + 3x_2 \\ y_2 = 3x_1 + 5x_2 \end{bmatrix}$

Then find the principle direction ie, the direction of the positive vector for which the direction vector of the position vector $y(Q)$ are excatly opposite. What does the boundary circle take under the deformation!

Soln:

Given $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ polar co-ordinate ⇒ To find the shape [$x = a\cos\theta$; $r = b\sin\theta$]

The characteristic equation of (2×2) matrix is $\lambda^2 - D_1\lambda + D_2 = 0$.

To find principle direction we need the angle, find angle ⇒ $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$, $\cos\beta = \frac{y}{\sqrt{x^2+y^2}}$ [x, y are eigen values]

$D_1 = 10$,

$D_2 = 16$.

The characteristic equation is $\lambda^2 - 10\lambda - 16 = 0$

The eigen values are $\lambda = 2, 8$

Eigen vectors:

$(A - \lambda I)X = 0$.

$$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (1).$$

When $\lambda = 2$ in (1)

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 + 3x_2 = 0$

$3x_1 + 3x_2 = 0$.

$\Rightarrow 3x_1 + 3x_2 = 0$

$3x_1 = -3x_2$

$x_1 = -x_2$

By assumption method.

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

When $\lambda = 8$ in (1).

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 3x_2 = 0$

$3x_1 - 3x_2 = 0$

$3x_1 = 3x_2$

$x_1 = x_2$

By assumption method

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$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We know that $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}$

when $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \cos\alpha = \frac{-1}{\sqrt{(-1)^2+(1)^2}}$

$$\alpha = \cos^{-1}\left(-1/\sqrt{2}\right)$$

$$\alpha = 135^\circ$$

when $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \cos^{-1} = \left[\frac{1}{\sqrt{1^2+1^2}}\right]$

$$\alpha = \cos^{-1}\left(1/\sqrt{2}\right)$$

$$\alpha = 45^\circ$$

Polar co-ordinates to find deformation.

The principle direction occurs in the angle 45° and 135°.

$x = a\cos\theta \qquad y = b\sin\theta$

Let $\lambda = 2, 8$

$X = 2\cos\theta$

$Y = 8\sin\theta$

$\cos\theta = x/2 \qquad \sin\theta = y/8$

we know that

$$\sin^2\theta + \cos^2\theta = 1$$

$$(x/2)^2 + (y/8)^2 = 1$$

$$x^2/2^2 + y^2/8^2 = 1$$

which is ellipse, The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$