2). Find the radius of the curve $y = e^{x}$ at $(0,1)$

at the pt. where it crosses the y axis

y axis $\Rightarrow x = 0$

$y = e^{x} \Rightarrow e^{0} = 1$

$\therefore (0,1)$

Soln.

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^{2}\right]^{3/2}}{\frac{d^{2}y}{dx^{2}}}$$

Given. $y = e^{x}$

$$\frac{dy}{dx} = e^{x}$$

At $(0,1)$, $\frac{dy}{dx} = 1$

and $\frac{d^{2}y}{dx^{2}} = e^{x}$

At $(0,1)$, $\frac{d^{2}y}{dx^{2}} = 1$

$$\therefore \rho = \frac{[1+1]^{3/2}}{1} = 2^{3/2} = 2\sqrt{2}$$

Hw 1]. Find the radius of curvature at $x = \frac{\pi}{2}$ on the curve $y = 4\sin x$ $\left(\frac{-1}{4}\right)$

2]. $y = c \log \sec\left(\frac{x}{c}\right)$ $(c\sec x/c)$

3]. Find radius of curvature of $xy = c^2$ at (c, c).

Soln.

Radius of curvature $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

Given. $xy = c^2 \rightarrow (1)$

Differentiate w.r. to 'x'

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x} \rightarrow (2)$$

$$\left(\frac{dy}{dx}\right)_{(c,c)} = -\frac{c}{c} = -1$$

Differentiate (2) w.r. to 'x'

$$\frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx} - y(1)}{x^2}\right] \qquad \because d\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$\left[\frac{d^2y}{dx^2}\right]_{(c,c)} = \frac{-[c(-1) - c]}{c^2} = \frac{c+c}{c^2}$$

$$= \frac{2c}{c^2} = \frac{2}{c}$$

$$\therefore \quad \rho = \frac{[1+(-1)^2]^{3/2}}{2/c}$$

$$= \frac{(1+1)^{3/2}}{2/c}$$

$$= 2^{3/2}\,\frac{c}{2}$$

$$= 2\sqrt{2}\,\frac{c}{2}$$

$$= c\sqrt{2}$$

$$\therefore \quad \rho = c\sqrt{2}.$$

$2^{3/2} = (2^{1/2})^3$

$= (\sqrt{2})^3$

$= 2\sqrt{2}$

H.W $xy = 30$ at $(3, 10)$

$\frac{(109)^{3/2}}{60}$

4]. Find the radius of curvature $(a/4, a/4)$ on $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

ie, $x^{1/2} + y^{1/2} = a^{1/2}$

Soln.

$\sqrt{x} + \sqrt{y} = \sqrt{a} \rightarrow (1)$

Differentiate (1) w.r.t. to 'x',

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

At $\left(\frac{a}{4}, \frac{a}{4}\right)$, $\quad \frac{dy}{dx} = \frac{-\sqrt{a/4}}{\sqrt{a/4}} = -1$

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\frac{dy}{dx} + \sqrt{y}\,\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

At $(a/4, a/4)$,

$$\frac{d^2y}{dx^2} = \frac{-\frac{\sqrt{a}}{2} \cdot \frac{1}{2\frac{\sqrt{a}}{2}}(-1) + \frac{\sqrt{a}}{2} \cdot \frac{1}{2\frac{\sqrt{a}}{2}}}{a/4}$$

$$= \frac{\frac{1}{2}+\frac{1}{2}}{a/4}$$

$$= \frac{4}{a}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+(-1)^2\right]^{3/2}}{4/a}$$

$$= \frac{2\sqrt{2}}{4/a}$$

$$= \frac{a}{\sqrt{2}}$$

5]. Find ρ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on $x^3+y^3 = 3axy$

Soln.

Gvn. $x^3+y^3 = 3axy \longrightarrow (1)$

Differentiate (1) w.r. to 'x'

$$3x^2+3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$x^2+y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$\frac{dy}{dx}\left[y^2-ax\right] = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $\frac{dy}{dx} = \dfrac{a \cdot \frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a \cdot \frac{3a}{2}}$

$$= \frac{6a^2 - 9a^2}{9a^2 - 6a^2}$$

$$= -1$$

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left[a\frac{dy}{dx} - 2x\right] - [ay - x^2]\left[2y\frac{dy}{dx} - a\right]}{(y^2 - ax)^2}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $\frac{d^2y}{dx^2} = -\frac{32}{3a}$

$$\rho = \frac{[1+1]^{3/2}}{-32/3a} = \frac{2\sqrt{2} \times 3a}{-32}$$

$$= -\frac{3\sqrt{2}\,a}{16}$$

ρ is +ve. $\therefore$ $\rho = \frac{3\sqrt{2}\,a}{16}$.

Q. Find the radius of curvature of $y = \frac{ax}{a+x}$ & hence show that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$

Soln.

Given. $y = \frac{ax}{a+x}$

$$\frac{dy}{dx} = \frac{(a+x)a - ax(1)}{(a+x)^2} = \frac{a^2}{(a+x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(a+x)^2 \cdot 0 - a^2 \cdot 2(a+x)}{(a+x)^4} = \frac{-2a^2}{(a+x)^3}$$

$$\rho = \frac{\left[1+\left[\frac{a^2}{(a+x)^2}\right]^2\right]^{3/2}}{-\frac{2a^2}{(a+x)^3}}$$

$$= \frac{\left[\frac{(a+x)^4+a^4}{(a+x)^4}\right]^{3/2}}{-\frac{2a^2}{(a+x)^3}} = \frac{[(a+x)^4+a^4]}{(a+x)^{4(\frac{3}{2})}} \times \frac{(a+x)^3}{-2a^2}$$

$$= \frac{[(a+x)^4+a^4]^{3/2}}{(a+x)^6} \times \frac{(a+x)^3}{-2a^2}$$

$$= \frac{[(a+x)^4+a^4]^{3/2}}{-2a^2(a+x)^3}$$

$$\therefore \rho = \frac{[(a+x)^4+a^4]^{3/2}}{2a^2(a+x)^3}$$

multiply by $\frac{2}{a}$

$$\frac{2\rho}{a} = \frac{2}{a} \cdot \frac{[(a+x)^4+a^4]^{3/2}}{2a^2(a+x)^3}$$

$$= \frac{[(a+x)^4+a^4]^{3/2}}{a^3(a+x)^3}$$

Taking power $2/3$

$$\left[\frac{2\rho}{a}\right]^{2/3} = \frac{[(a+x)^4+a^4]}{a^2(a+x)^2}$$

$$= \frac{(a+x)^4}{a^2(a+x)^2} + \frac{a^4}{a^2(a+x)^2}$$

$$= \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2}$$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2} \qquad \therefore \left[\frac{2\rho}{a}\right]^{2/3} = \left[\frac{y}{x}\right]^2 + \left[\frac{x}{y}\right]^2$$

7. Find the radius of curvature at $(-2, 0)$ on $y^2 = x^3 + 8$.

Soln.

Given. $y^2 = x^3 + 8$

$$2y \frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

At $(-2, 0)$, $\frac{dy}{dx} = \infty$

Now, $y^2 = x^3 + 8$

$$\Rightarrow x^3 = y^2 - 8$$

$$3x^2 \frac{dx}{dy} = 2y$$

$$\frac{dx}{dy} = \frac{2y}{3x^2}$$

At $(-2, 0)$, $\frac{dx}{dy} = 0$

$$\frac{d^2x}{dy^2} = \frac{3x^2(2) - 2y \cdot 6x \cdot \frac{dx}{dy}}{9x^4}$$

$$= \frac{6x^2 - 12xy\,\frac{dx}{dy}}{9x^4}$$

At $(-2,0)$, $\quad \dfrac{d^2x}{dy^2} = \dfrac{6(4)-0}{9(-2)^4} = \dfrac{24}{9\times16} = \dfrac{1}{6}$

WKT, $\quad \rho = \dfrac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$

$$\therefore \rho = \frac{(1+0)^{3/2}}{1/6}$$

$$\boxed{\rho = 6}$$

1]. Find the radius of curvature of $y = \cosh\left(\frac{x}{c}\right)$

Soln.

Given. $y = \cosh\left(\frac{x}{c}\right)$

$$y_1 = \sinh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

$$y_2 = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c^2}$$

$$\therefore \rho = \frac{\left\{1 + \left[\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{c}\right]^2\right\}^{3/2}}{\frac{1}{c^2}\cdot\cosh\left(\frac{x}{c}\right)}$$

$$\rho = \frac{\left[1 + \sinh^2\left(\frac{x}{c}\right)\frac{1}{c^2}\right]^{3/2}}{\frac{1}{c^2}\cosh\left(\frac{x}{c}\right)}$$