Envelope :

The envelope of the family of curves is a curve which touches each membrane of a family.

Procedure to find the envelope of the family of curves:

Method 1:

If the family of curves is expressed as the quadratic form say $Aa^2 + Ba + C = 0$ where 'a' is the parameter. Then the envelope is given by

$$B^2 - 4AC = 0$$

Method 2:

* Differentiate given eqn. with respect to parameter.
* Eliminate the parameter from the given curve.

Problems based on envelope with one parameter

1. Find the envelope of $y = mx + am^2$, m is parameter

Soln.

The given eqn. is $y = mx + am^2$ (m is parameter)

i.e., $am^2 + xm - y = 0$, which is a quadratic eqn. in m.

The envelope is given by,

$$B^2 - 4AC = 0$$

Here $A = a, B = x, C = -y$

$$\therefore x^2 - 4a(-y) = 0$$

$$x^2 + 4ay = 0$$

2. Find the envelope of $y = mx + \sqrt{a^2m^2 + b^2}$

Soln.

Gvn. $y = mx + \sqrt{a^2m^2 + b^2} \Rightarrow y - mx = \sqrt{a^2m^2 + b^2}$

Squaring on both sides,

$$(y - mx)^2 = a^2m^2 + b^2$$

$$\Rightarrow y^2 - 2mxy + m^2 x^2 - a^2 m^2 - b^2 = 0$$

$$(x^2 - a^2) m^2 - 2(xy) m + (y^2 - b^2) = 0 \text{ which is a quadratic eqn. in } m.$$

The envelope is gvn. by $\quad B^2 - 4AC = 0$

$$\text{Here } A = (x^2 - a^2), \quad B = -2xy \quad C = y^2 - b^2$$

$$4x^2 y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$\div 4 \qquad x^2 y^2 = (x^2 - a^2)(y^2 - b^2)$$

3). Find the envelope of $\dfrac{x^2}{\alpha} + \dfrac{y^2}{1-\alpha} = 1$, where α is the parameter

Soln.

$$\text{Gvn. } \frac{x^2}{\alpha} + \frac{y^2}{1-\alpha} = 1$$

$$(1-\alpha) x^2 + \alpha y^2 = \alpha(1-\alpha)$$

$$\Rightarrow (1-\alpha) x^2 + \alpha y^2 - \alpha + \alpha^2 = 0$$

$$\Rightarrow \alpha^2 + x^2 - \alpha x^2 + \alpha y^2 - \alpha = 0$$

$$\Rightarrow \alpha^2 + (-x^2 + y^2 - 1)\alpha + x^2 = 0$$

$$\text{Here } A = 1, \quad B = (-x^2 + y^2 - 1), \quad C = x^2$$

$$\text{Now } \quad B^2 - 4AC = 0.$$

$$[-x^2 + y^2 - 1]^2 - 4(1)(x^2) = 0$$

$$x^4 + y^4 + 1 - 2x^2 y^2 - 2y^2 + 2x^2 - 4x^2 = 0$$

$$x^4 + y^4 + 1 - 2x^2 y^2 - 2y^2 - 2x^2 = 0$$

4]. Find the envelope of $x\cos\theta + y\sin\theta = a$, where θ is the parameter.

Soln.

$$\text{Gvn. } \quad x\cos\theta + y\sin\theta = a \quad \longrightarrow (1)$$

Differentiate partially w.r. to 'θ',

$$-x\sin\theta + y\cos\theta = 0 \quad \longrightarrow (2)$$

Squaring of (1) and (2) separately,

$$[x\cos\theta + y\sin\theta]^2 + [-x\sin\theta + y\cos\theta]^2 = a^2 + 0$$

$$x^2 \cos^2\theta + y^2 \sin^2\theta + 2xy\sin\theta\cos\theta + x^2\sin^2\theta + y^2\cos^2\theta$$
$$- 2xy\sin\theta\cos\theta = a^2$$

$$x^2[\sin^2\theta + \cos^2\theta] + y^2[\sin^2\theta + \cos^2\theta] = a^2$$

$$\Rightarrow \quad x^2(1) + y^2(1) = a^2$$
$$x^2 + y^2 = a^2$$