Eigen Values and Eigen Vectors of a real matrix :

Let $A = [a_{ij}]$ be a square matrix.

The characteristic equation of A is $|A - \lambda I| = 0$.

The roots of the characteristic equation are called eigen values of A.

If there exists a non-zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, such that $AX = \lambda X$, then the vector X is called an eigen vector of A corresponding to the eigen value of λ.

Note :

Let A be a square matrix. Then

* Sum of the eigen values of A = Sum of the main diagonal elements of A.
* Product of the eigen values of A = Determinant of A.
* If $(\lambda_1, \lambda_2, \cdots \lambda_n)$ are the eigen values of A, then $(\lambda_1^n, \lambda_2^n, \cdots \lambda_n^n)$ are the eigen values of A^n.
* If $(\lambda_1, \lambda_2, \cdots \lambda_n)$ are the eigen values of A, then $(k\lambda_1, k\lambda_2, \cdots k\lambda_n)$ are the eigen values of kA.
* If λ is an eigen value of a non-singular matrix A, then
  (i) λ^{-1} is an eigen value of A^{-1}.
  (ii) $\frac{|A|}{\lambda}$ is an eigen value of adj A.

(3)

* A square matrix A and its transpose A^T have the same eigen values.
* The eigen values of a triangular matrix are just the main diagonal elements of the matrix.
* The eigen values of a symmetric matrix are real numbers.
* The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

Type I :

Symmetric or Non-symmetric matrices with non-repeated eigen values :

(1) Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

Soln: Let $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Step 1 : To find the characteristic equations :

The characteristic equation is,

$$\lambda^3 - c_1 \lambda^2 + c_2 \lambda - c_3 = 0$$

$$c_1 = 7 + 6 + 5 = 18$$

$$C_2 = \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$$

$$C_2 = 99$$

$$C_3 = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix} = 162$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$$

Step 2 : To find the eigen values :

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \longrightarrow (1)$$

$$\boxed{\lambda = 3, 6, 9}$$

Step 3 : To find the eigen vectors :

$$(A - \lambda I)X = 0.$$

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Case (i) : $\lambda = 3$

$$(2) \Rightarrow \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 4 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix}}$$

$$\begin{matrix} 4 & & -2 & & 0 & & 4 \\ & \times & & \times & & \times & \\ -2 & & 3 & & -2 & & -2 \end{matrix}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{8}$$

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$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Case (ii) : $\lambda = 6$

$$② \Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} 1 & -2 & 0 & 1 \\ -2 & 0 & -2 & -2 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ -2 & 0 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}.$$

$\therefore$ The eigen Vector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Case (iii) : $\lambda = 9$

$$② \Rightarrow \begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 0 \\ -3 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -3 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2}$$

Hence the eigen values and eigen vectors are

λ : 3, 6, 9

X : $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

(2) Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

Soln:

$\lambda = 1, 2, 3$

$X = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

(3) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ Soln: $\lambda_1 = 1, -1, 4$.

$X = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(4) $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ Soln: $\lambda = 1, 3, -4$

$X = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$

(5) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ Soln: $\lambda = -2, 3, 6$

$X = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$